A new class of symmetry preserving and thermodynamically consistent subgrid-scale models

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Outline

- Motivation
- Symmetry
- Symmetry and analysis of turbulence models
- New SGS models (symmetry + thermodynamics)
- Numerical test

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Motivation

Turbulence models → Respect of fundamental properties of the flow

- Conservation laws
- Scaling laws
- Spectral properties
- Self-similar solutions
- Etc

SYMMETRY

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Symmetries $\Longleftrightarrow$ Physics of the equations

- Transformation :

$$NS(t, x, u, p) = 0 \quad \longmapsto \quad NS(\hat{t}, \hat{x}, \hat{u}, \hat{p}) = 0$$

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- Symmetry $\xrightarrow{\text{Nœther}}$ Conservation law

Temporal translations	$\longrightarrow$	*Energy*
Spatial translations	$\longrightarrow$	*Linear momentum*
Rotations	$\longrightarrow$	*Angular momentum*

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- Symmetries $\xrightarrow{\text{Oberlack}}$ Scaling laws (wall laws, ...)
- Symmetries $\xrightarrow{\text{Ünal}}$ Cascade of Kolmogorov
- Symmetries $\xrightarrow{\text{Grassi } et\ al.}$ Vortex solutions
- ...

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Symmetry group of NS equations

- Time translations : $(t, x, u, p) \mapsto (t + a, x, u, p)$
- Pressure translations : $(t, x, u, p) \mapsto (t, x, u, p + \zeta(t))$
- Rotations : $(t, x, u, p) \mapsto (t, Rx, Ru, p)$
- Generalized Galilean transformations :

$$(t, x, u, p) \mapsto (t,\ x + \alpha(t),\ u + \alpha'(t),\ p - \rho x \cdot \alpha''(t))$$

fdy

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- Second scale transformations :
$$(t, x, u, p, \nu) \mapsto (t,\ ax,\ au,\ a^2 p,\ a^2 \nu)$$

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- Others : reflections, 2D material indifference

Turbulence model and symmetry

- Respect of the physics of the flow ⟷ Respect of symmetries

- Introduction of a turbulence model

  Navier–Stokes (NS) $\longrightarrow$ G-NS=$\{$symmetries of NS$\}$

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NS + Turb. model $\longrightarrow$ G-NSM

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G-NS $\subset$ G-NSM

↓

Conservation laws, scaling laws, ...

Analysis of some SGS models

	Translations	Rotations	Scale changes	2D material indifference
Smagorinsky	Yes	Yes	*No*	Yes
Dynamic	Yes	Yes	Yes	Yes
Structure	Yes	Yes	*No*	*No*
Gradient	Yes	Yes	*No*	*No*
Taylor	Yes	Yes	*No*	*No*
Rational	Yes	Yes	*No*	*No*
Similarity	Yes	Yes	Yes	Yes
Lund	Yes	Yes	*No*	*No*
Kosovic	Yes	Yes	*No*	*No*

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! ! ! Second law of thermodynamics

→ Other models

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Symmetry consistent SGS models

- (Time, pressure and Galilean) translations and rotations

$$\tau_s = \mathcal{F}(\overline{\mathsf{S}}, q, \epsilon)$$

τ_s : SGS model, $\overline{\mathsf{S}}$: (filtered) strain rate tensor

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- Invariance theory

$$\tau_s^d = A(\chi, \zeta, q, \epsilon)\,\overline{\mathsf{S}} + B(\chi, \zeta, q, \epsilon)\ \mathrm{Adj}^d\,\overline{\mathsf{S}}$$

Deviatoric : $\mathsf{M}^d = \mathsf{M} - \frac{1}{3}(\operatorname{tr}\mathsf{M})\mathsf{I_d}$

A and B are scalar arbitrary functions

Invariants of $\overline{\mathsf{S}}$: $\chi = \|\overline{\mathsf{S}}\|^2$, $\zeta = \det\overline{\mathsf{S}}$

Adjoint or comatrix : $(\mathrm{Adj}\,\overline{\mathsf{S}})\overline{\mathsf{S}} = (\det\overline{\mathsf{S}})\mathsf{I_d}$

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Symmetry consistent models

- Scale transformations

$$\tau_s^d = \frac{q^2}{\epsilon} \left(A_1(v)\, \overline{\mathsf{S}} + \frac{1}{\sqrt{\chi}} B_1(v) \;\; \mathrm{Adj}^d\, \overline{\mathsf{S}} \right)$$

$$v = \frac{\zeta}{\chi^{3/2}} = \frac{\det \overline{\mathsf{S}}}{||\overline{\mathsf{S}}||^{3/2}}$$

→ Class of SGS models consistent with the symmetries of NS

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Second law of thermodynamics

- Viscous stress tensor : $\tau = 2\nu \mathsf{S} = \dfrac{\partial \psi}{\partial \mathsf{S}}$

  "potential" $\psi = \nu \ \mathrm{tr}\, \mathsf{S}^2$

- ψ positive and convex $\Rightarrow$ positive dissipation :

$$\Phi = \ \mathrm{tr}(\tau \ \mathsf{S}) \geq 0$$

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- τ_s subgrid tensor,

  dissipation : $\Phi_s = \mathrm{tr}(\tau_s\, \overline{\mathsf{S}})$

- Hypothesis : $\tau_s = \dfrac{\partial \psi_s}{\partial \overline{\mathsf{S}}}$

Consequences on the model

- Consequence of the “potential” hypothesis :

$$\tau_s^d = \frac{q^2}{\varepsilon}\left[\Big(2g(v) - 3v\dot{g}(v)\Big)\overline{\mathsf{S}} + \frac{1}{||\overline{\mathsf{S}}||}\,\dot{g}(v)\,\mathrm{Adj}^d\,\overline{\mathsf{S}}\right],$$

g : arbitrary function

- Positive total dissipation

$$\overline{\Phi} + \Phi_s \geq 0 \qquad \Leftrightarrow \qquad \nu + \frac{q^2}{\varepsilon} g \geq 0$$

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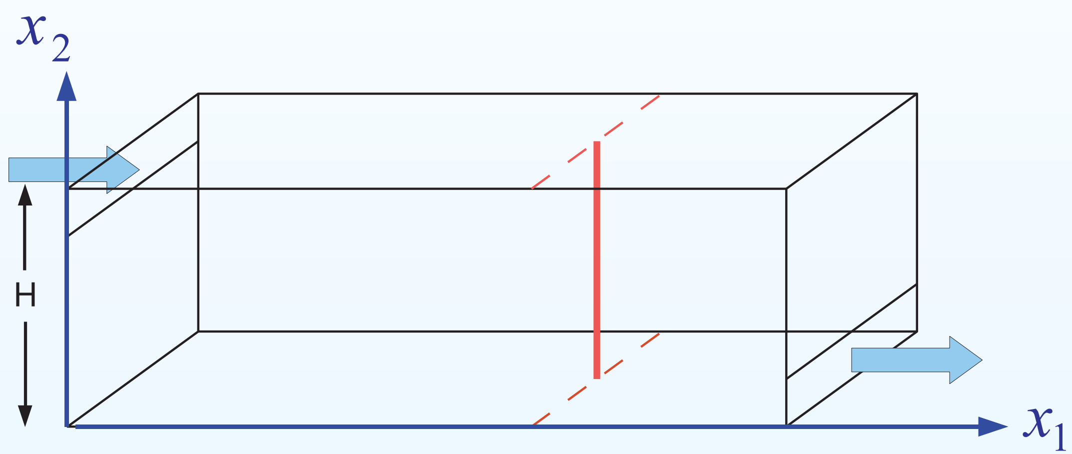
Numerical test

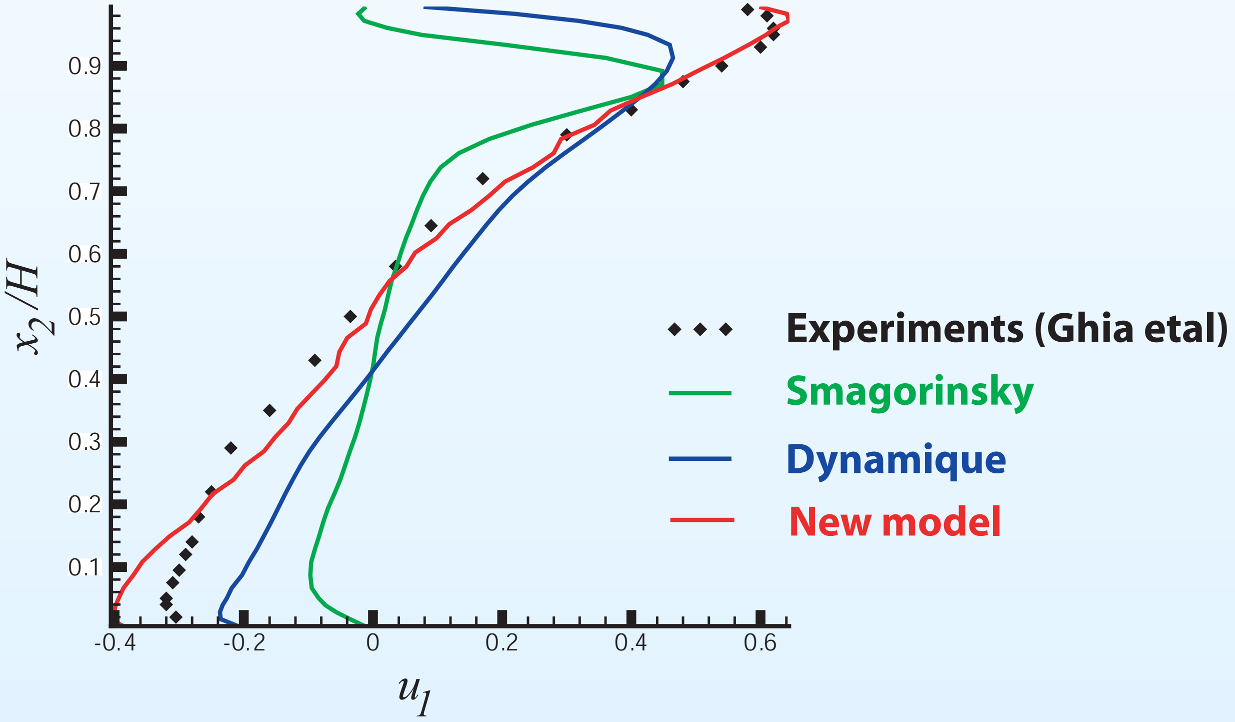
- 3D ventilated room (Nielsen's cavity, Re $\simeq$ 5000)



Numerical test

- 3D ventilated room (Nielsen's cavity, Re $\simeq$ 5000)

- Profile of the mean horizontal velocity



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Conclusion

- Symmetry → physically consistent SGS models

  → coherent with experiments

Conclusion

- Symmetry → physically consistent SGS models
  → coherent with experiments

- Open question : the parameters of the model
  - Dynamical procedure
  - Homogeneous turbulence approximations
  - Self-similar solutions

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