New Criteria for the Eduction of Three-Dimensional Turbulent Structures

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multiple definitions of a vortex $\Rightarrow$ multiple criteria

- spiraling (locally linearized) trajectories
  Chong, Perry & Cantwell '90: Δ
  Chakraborty, Balachandar & Adrian '05
- balance between shear and rotation
  Hunt, Wray & Moin '88: Q
- approximated local minima of pressure
  Jeong & Hussain '95: λ_2
- combination of definitions
  Kida & Miura '98
  Horiuti '01

- incompressible flows: Δ, Q, $\lambda_2 \rightarrow$ (Okubo–) Weiss '81, '91 criterion
- Weiss criterion recasted using streamfunction ψ

$$\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial x^2} - \left(\frac{\partial^2\psi}{\partial x\,\partial y}\right)^2 > 0 \qquad \Leftrightarrow \qquad \text{convex streamlines}$$

- alternative definition: region of convex isovorticity lines

$$\frac{\partial^2\omega}{\partial x^2}\frac{\partial^2\omega}{\partial x^2} - \left(\frac{\partial^2\omega}{\partial x\,\partial y}\right)^2 > 0$$

Herbert, Larchevêque & Staquet '96

- advantages:
  - educe thinner structures than Weiss criterion
  - well correlated with dynamically active regions of the flow

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Introduction

Geometrical interpretation for 2D flows

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Mathematical definitions
Curvature in 3D space (1)

- local analysis of isosurfaces of the vorticity norm $\omega = \left\|\underline{\omega}\right\|$
- Taylor expansion with M and M_0 on the same isosurface

$$\frac{1}{2}\,\underline{MM_0}^{\mathrm{T}}\;\underline{\underline{\mathrm{H}}}(\omega)\Big|_{M_0}\;\underline{MM_0} \quad + \quad \underline{\nabla}(\omega)\Big|_{M_0}\cdot\underline{MM_0} \quad = \quad \mathcal{O}\left(\left\|\underline{MM_0}\right\|^3\right)$$

- Hessian matrix and gradient vector
  - $\underline{\underline{\mathrm{H}}}(\omega)$ symmetric → diagonalizable:

$$\underline{\underline{\mathrm{H}}}(\omega)_{\left(\underline{e_1},\underline{e_2},\underline{e_2}\right)} = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$$

  - gradient in the frame $\left(\underline{e_1},\underline{e_2},\underline{e_3}\right)$:

$$\underline{\nabla}(\omega)\Big|_{M_0} = a_1\,\underline{e_1} + a_2\,\underline{e_2} + a_3\,\underline{e_3}$$

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Mathematical definitions
Curvature in 3D space (2)

- S locally described in the frame $\left(M_0, \underline{e_1}, \underline{e_2}, \underline{e_3}\right)$ by:

$$\lambda_1\left(X_1+\frac{a_1}{\lambda_1}\right)^2+\lambda_2\left(X_2+\frac{a_2}{\lambda_2}\right)^2+\lambda_3\left(X_3+\frac{a_3}{\lambda_3}\right)^2=\frac{a_1^2}{\lambda_1}+\frac{a_2^2}{\lambda_2}+\frac{a_3^2}{\lambda_3}$$

Quadric with axis $\left(\underline{e_1}, \underline{e_2}, \underline{e_3}\right)$ centered on $C:\left(\frac{-a_1}{\lambda_1}, \frac{-a_2}{\lambda_2}, \frac{-a_3}{\lambda_3}\right)$

- exact mean curvature $\mathcal{H}$ at location M_0

$$\mathcal{H}=\frac{\underline{\nabla}(\omega)_{|M_0}^{T}\ \underline{\underline{H}}(\omega)_{|M_0}\ \underline{\nabla}(\omega)_{|M_0}-\mathrm{Tr}\left(\underline{\underline{H}}(\omega)_{|M_0}\right)\left\|\underline{\nabla}(\omega)_{|M_0}\right\|^2}{2\left\|\underline{\nabla}(\omega)_{|M_0}\right\|^3}$$

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Curvature in 3D space (2)

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$$\lambda_1 \left(X_1 + \frac{a_1}{\lambda_1}\right)^2 + \lambda_2 \left(X_2 + \frac{a_2}{\lambda_2}\right)^2 + \lambda_3 \left(X_3 + \frac{a_3}{\lambda_3}\right)^2 = \frac{a_1^2}{\lambda_1} + \frac{a_2^2}{\lambda_2} + \frac{a_3^2}{\lambda_3}$$

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$$\mathcal{H} = \frac{\underline{\nabla}(\omega)_{|M_0}^{T} \; \underline{\underline{H}}(\omega)_{|M_0} \; \underline{\nabla}(\omega)_{|M_0} - \mathrm{Tr}\left(\underline{\underline{H}}(\omega)_{|M_0}\right) \left\|\underline{\nabla}(\omega)_{|M_0}\right\|^2}{2 \left\|\underline{\nabla}(\omega)_{|M_0}\right\|^3}$$

- using the eigenvalues
  - 3 negative eigenvalues: $\begin{cases} \text{ellipsoid} \\ \text{maxima of } \omega \end{cases}$
  - 2 negative eigenvalues: $\begin{cases} \text{hyperboloid} \\ \text{saddle point maxima of } \omega \end{cases}$
- using the main invariants

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$$\begin{cases} I_1 = \mathrm{Tr}\left[\underline{\underline{\mathrm{H}}}(\omega)\right] = \lambda_1 + \lambda_2 + \lambda_3 \\ I_2 = \frac{1}{2}\left\{\mathrm{Tr}\left[\underline{\underline{\mathrm{H}}}(\omega)\right]^2 - \mathrm{Tr}\left[\underline{\underline{\mathrm{H}}}(\omega)^2\right]\right\} = \lambda_1\,\lambda_2 + \lambda_1\,\lambda_3 + \lambda_2\,\lambda_3 \\ I_3 = \mathrm{Det}\left[\underline{\underline{\mathrm{H}}}(\omega)\right] = \lambda_1\,\lambda_2\,\lambda_3 \end{cases}$$

Mathematical definitions
Classification

- using the eigenvalues
  - 3 negative eigenvalues: $\begin{cases} \text{ellipsoid} \\ \text{maxima of } \omega \end{cases}$
  - 2 negative eigenvalues: $\begin{cases} \text{hyperboloid} \\ \text{saddle point maxima of } \omega \end{cases}$
- using the main invariants
  - $\left.\begin{array}{ccc} I_1 & < & 0 \\ I_2 & > & 0 \\ I_3 & < & 0 \end{array}\right\}:$ $\begin{cases} \text{ellipsoid} \\ \text{maxima of } \omega \end{cases}$
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  - $\left.\begin{array}{lcl} I_1 & < & 0 \\ I_3 & > & 0 \end{array}\right\}$: $\begin{cases} \text{hyperboloid} \\ \text{saddle point maxima of } \omega \end{cases}$ $\rightarrow$ tube / layer

Mathematical definitions

Refined classification

- separating tube and layer / sheet
  - slenderness governed by I_2: $\quad I_2 > 0 \quad \Leftrightarrow \quad \lambda_3 < \dfrac{\lambda_1 \lambda_2}{|\lambda_1 + \lambda_2|}$

$I_2 > 0 \quad \rightarrow \quad$ tube ($\mathrm{H}_\omega^{\mathrm{t}}$)

$I_2 \leq 0 \quad \rightarrow \quad$ layer / sheet

- separating rotational layer and sheet
  - difficult to achieve using invariants

physical consideration	geometrical consideration
$Q > 0 \quad \rightarrow$ rotational layer ($\mathrm{H}_\omega^{\mathrm{rl}\,Q}$)	$\mathcal{H} > 0 \quad \rightarrow$ rotational layer ($\mathrm{H}_\omega^{\mathrm{rl}\,\mathcal{H}}$)
$Q \leq 0 \quad \rightarrow$ sheet ($\mathrm{H}_\omega^{\mathrm{s}\,Q}$)	$\mathcal{H} \leq 0 \quad \rightarrow$ sheet ($\mathrm{H}_\omega^{\mathrm{s}\,\mathcal{H}}$)

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geometrical consideration

$\mathcal{H} > 0 \quad \rightarrow$ rotational layer $(\mathrm{H}^{\mathrm{rl}\,\mathcal{H}}_{\omega})$

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$$\mathcal{Q} > 0 \quad \rightarrow \quad \text{rotational layer} \quad (\mathrm{H}^{\mathrm{rl}\,\mathcal{Q}}_{\omega})$$

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geometrical consideration

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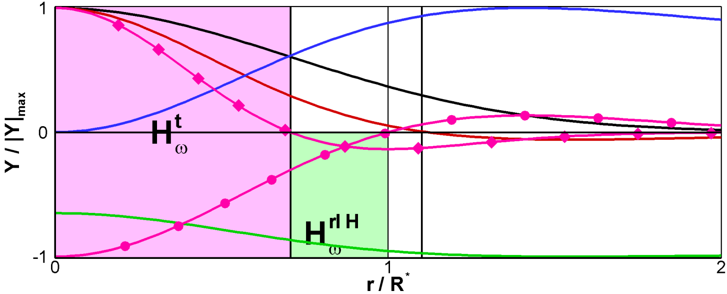
Test cases

Burgers' vortex tube

Re=1



ω: black line　Δ: red line　Q: green line　λ_2: blue line

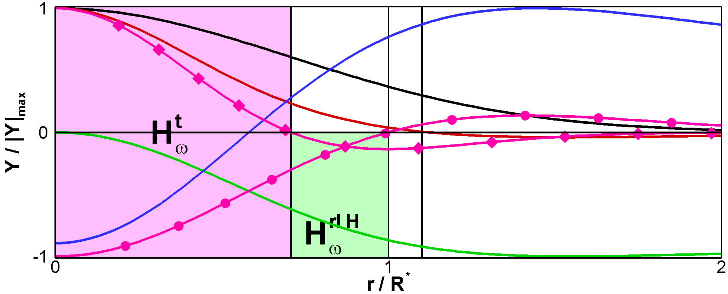
I_1: line with circles　I_2: line with diamonds　$I_3 = 0$　$\mathcal{H} = \frac{1}{r} > 0$

- H_ω criteria relying on $\mathcal{H}$: insensitive to transverse shear
- $I_1 \leq 0$ corresponds to the "classical" definition of the vortex core:

$$r \leq R^* = 2\sqrt{\frac{\nu}{\alpha}}$$

Test cases
Burgers' vortex tube

Re= $\sqrt{3}$



ω: ——— Δ: ——— Q: ——— λ_2: ———

I_1: —●— I_2: —◆— $I_3 = 0$ $\mathcal{H} = \frac{1}{r} > 0$

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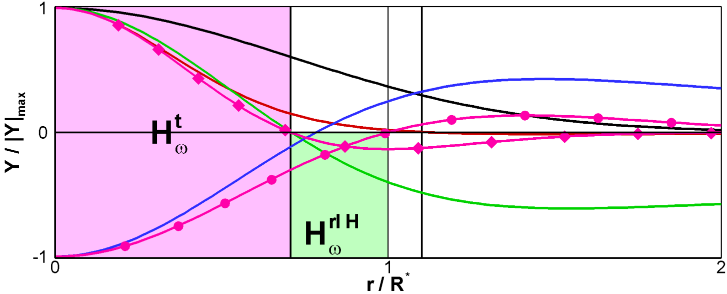
Test cases

Burgers' vortex tube

Re=3



ω: ——— Δ: ——— Q: ——— λ_2: ———

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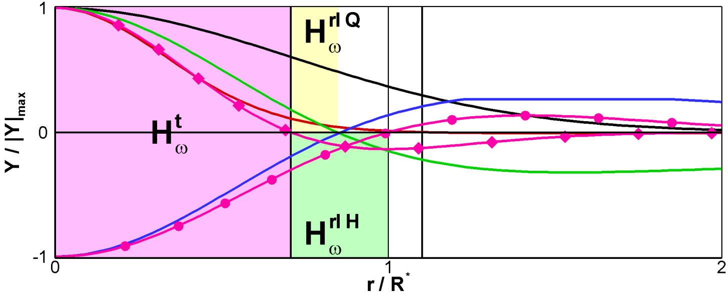
Test cases

Burgers' vortex tube

Re=4



ω: —— Δ: —— Q: —— λ_2: ——

I_1: —●— I_2: —◆— $I_3 = 0$ $\mathcal{H} = \frac{1}{r} > 0$

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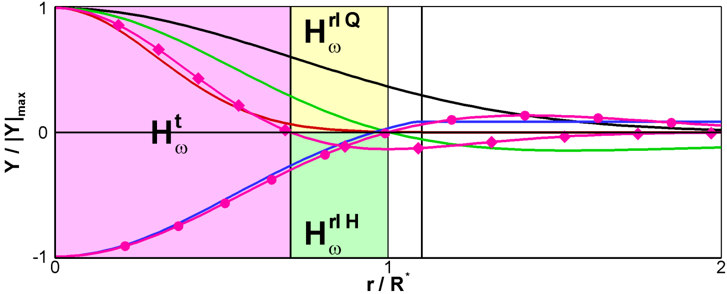
Test cases
Burgers' vortex tube

Re=$4\sqrt{3}$



ω: — Δ: — Q: — λ_2: —

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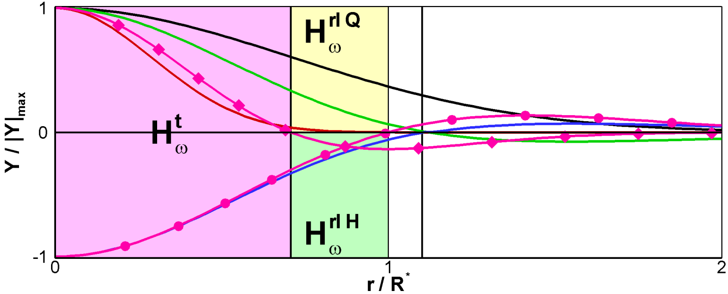
Test cases
Burgers' vortex tube

Re=100



ω: —— Δ: —— Q: —— λ_2: ——

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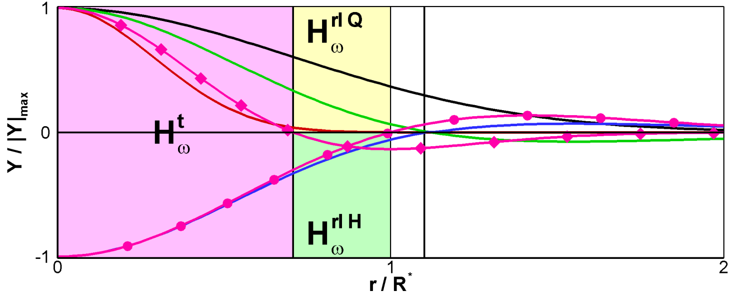

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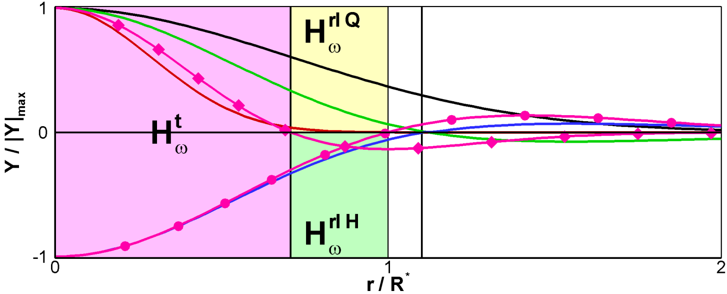

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$$\left\{\begin{array}{rcl} \underline{u} & = & U(x_2)\,\underline{i_1} - \alpha\, x_2\,\underline{i_2} + \alpha\, x_3\,\underline{i_3} \\ \\ \omega & = & -U'(x_2) = \dfrac{\gamma}{\sqrt{2\pi}}\sqrt{\dfrac{\alpha}{\nu}}\,\exp\left(\dfrac{-\alpha\, x_2^2}{2\nu}\right) \end{array}\right.$$

- $\Delta = \alpha^6 \quad > \quad 0$
- $Q = -\alpha^2 \quad < \quad 0$
- $\lambda_2 = \dfrac{1}{2}\left[\alpha^2 + \sqrt{\alpha^2\left(1 + U'(x_2)^2\right)}\right] \quad > \quad 0$

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- $\Delta = \alpha^6 \quad > \quad 0$
- $Q = -\alpha^2 \quad < \quad 0$
- $\lambda_2 = \dfrac{1}{2} \left[\alpha^2 + \sqrt{\alpha^2 \left(1 + U'(x_2)^2 \right)} \right] \quad > \quad 0$

The three criteria exclude the whole layer

Test cases

Burgers' vortex layer

$$\begin{cases} \underline{u} & = & U(x_2)\ \underline{i_1} - \alpha\, x_2\, \underline{i_2} + \alpha\, x_3\, \underline{i_3} \\ \\ \omega & = & -U'(x_2) = \dfrac{\gamma}{\sqrt{2\pi}} \sqrt{\dfrac{\alpha}{\nu}}\ \exp\left(\dfrac{-\alpha\, x_2^2}{2\nu} \right) \end{cases}$$

- $I_1 = -\dfrac{\gamma}{\sqrt{2\pi}} \left(\dfrac{\alpha}{\nu}\right)^{\frac{3}{2}} \exp\left(\dfrac{-\alpha\, x_2^2}{2\nu} \right) \left(1 - \dfrac{\alpha}{\nu} x_2^2\right) \qquad I_2 = I_3 = \mathcal{H} = 0$

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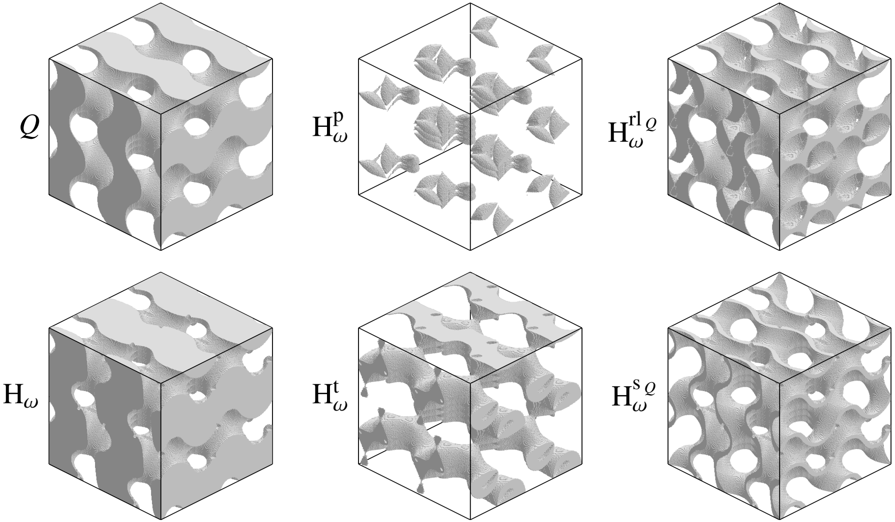
Burgers' vortex layer

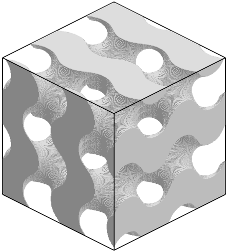
$$\left\{ \begin{array}{rcl} \underline{u} & = & U(x_2)\ \underline{i_1} - \alpha\, x_2\, \underline{i_2} + \alpha\, x_3\, \underline{i_3} \\ \\ \omega & = & -U'(x_2) = \dfrac{\gamma}{\sqrt{2\pi}} \sqrt{\dfrac{\alpha}{\nu}}\ \exp\left(\dfrac{-\alpha\, x_2^2}{2\nu} \right) \end{array} \right.$$

- $I_1 = -\dfrac{\gamma}{\sqrt{2\pi}} \left(\dfrac{\alpha}{\nu}\right)^{\frac{3}{2}} \exp\left(\dfrac{-\alpha\, x_2^2}{2\nu}\right) \left(1 - \dfrac{\alpha}{\nu} x_2^2\right) \qquad I_2 = I_3 = \mathcal{H} = 0$
- $I_1 < 0 \quad \Leftrightarrow \quad |x_2| < \delta \equiv \sqrt{\dfrac{\nu}{\alpha}}$

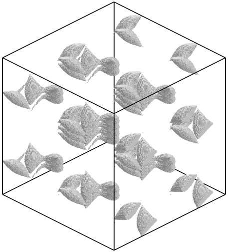
Both $\mathrm{H}_\omega^{\mathrm{S}\,Q}$ and $\mathrm{H}_\omega^{\mathrm{S}\,\mathcal{H}}$ criteria educe the whole layer


Q
H_ω^{p}
$H_\omega^{rl\,Q}$
H_ω
H_ω^{t}
$H_\omega^{s\,Q}$

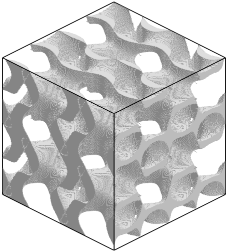
Q



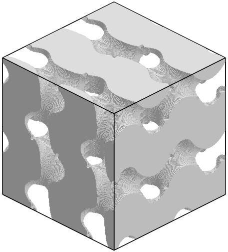
$\mathrm{H}_\omega^{\mathrm{p}}$



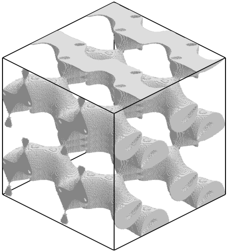
$\mathrm{H}_\omega^{\mathrm{rl}\,\mathcal{H}}$



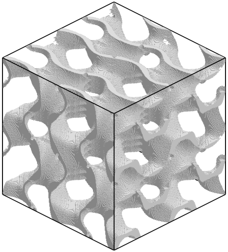
H_ω



$\mathrm{H}_\omega^{\mathrm{t}}$



$\mathrm{H}_\omega^{\mathrm{s}\,\mathcal{H}}$



Test cases

Forced isotropic turbulence (Vincent & Meneguzzi, JFM, 1991)

- Q, λ_2 and H_ω regions: $\simeq$ 40 % of the total volume
- conditional averaging of $\begin{cases} k,\ \epsilon,\ \Omega^2/\mathrm{S}^2 \\ \text{terms of the enstrophy equation} \end{cases}$

- differences between H_ω and $(Q,\ \lambda_2)$ regions
  - higher ϵ (inverse trend)
  - enstrophy equation: $\begin{cases} \text{lower} & \mathrm{dis}_{\omega^2} \\ \text{higher} & \mathrm{P}_{\omega^2},\ \mathrm{dif}_{\omega^2},\ \mathrm{D}_{\omega^2} \end{cases}$
  - higher V.S.R. (inverse trend)

- subdivisions of H_ω: $\mathrm{H}_\omega^{\mathrm{p}}$ (6 %) $\mathrm{H}_\omega^{\mathrm{t}}$ (11 %) $\mathrm{H}_\omega^{\mathrm{rl}\,\mathcal{H}}$ (20 %) $\mathrm{H}_\omega^{\mathrm{s}\,\mathcal{H}}$ (4 %)
  - $\mathrm{H}_\omega^{\mathrm{p}} \rightarrow \mathrm{H}_\omega^{\mathrm{t}} \rightarrow \mathrm{H}_\omega^{\mathrm{rl}\,\mathcal{H}} \rightarrow \mathrm{H}_\omega^{\mathrm{s}\,\mathcal{H}}$: ω^2, P_{ω^2}, dis_{ω^2}, dif_{ω^2}, D_{ω^2}, Ω^2/S^2 $\searrow$
  - $\left(\mathrm{H}_\omega^{\mathrm{p}}, \mathrm{H}_\omega^{\mathrm{t}}\right) \rightarrow \left(\mathrm{H}_\omega^{\mathrm{rl}\,\mathcal{H}}, \mathrm{H}_\omega^{\mathrm{s}\,\mathcal{H}}\right)$: ϵ, V.S.R. $\searrow$

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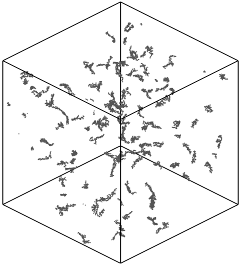
Test cases

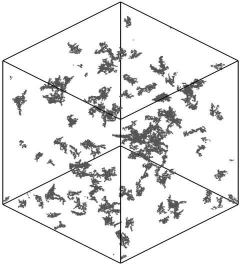
Freely decaying homogeneous turbulence (Liechtenstein, Godeferd & Cambon, JoT, 2005)

Isotropic

$\mathrm{H}^{\mathrm{p}}_{\omega}$



$\mathrm{H}^{\mathrm{t}}_{\omega}$



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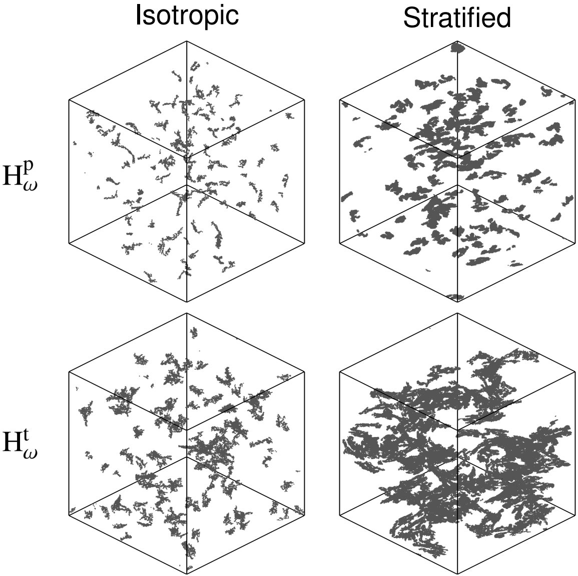

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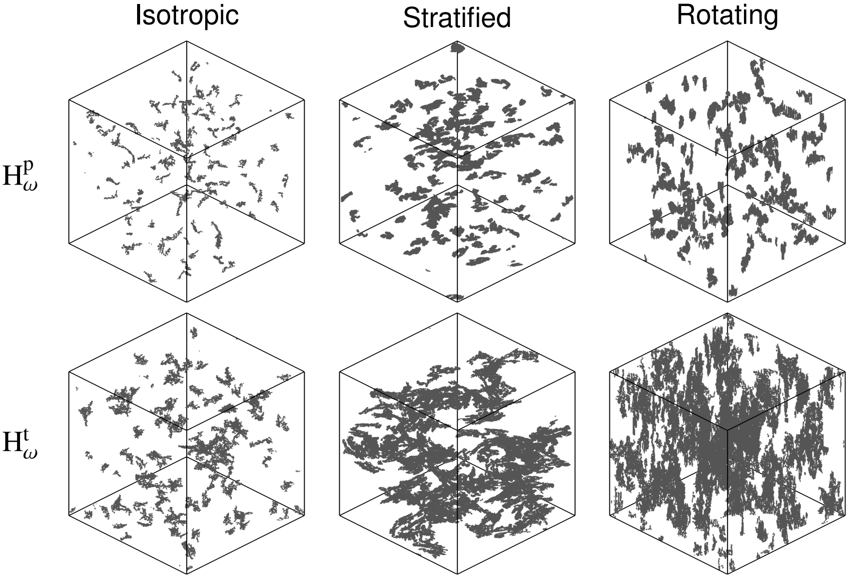

Conclusion and future work

- geometrically defined criteria ...
- ... also related to maxima of ω

- Galilean invariant
- educe low percentages of the total volume ...
- ... without any arbitrary threshold

- educe small scales structures ...
- ... with specific enstrophy dynamics: $\left\{ \begin{array}{c} \text{high vortex stretching} \\ \text{high diffusion} \\ \text{low dissipation} \end{array} \right\}$ ...
- ... complementary to structures defined by classical criteria

- to be applied to other flowfields
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Definitions of the members of the H_ω family

	$\mathrm{H}_\omega^{\mathrm{p}}$	$\mathrm{H}_\omega^{\mathrm{t}}$	$\mathrm{H}_\omega^{\mathrm{rl}\,Q}$	$\mathrm{H}_\omega^{\mathrm{s}\,Q}$	$\mathrm{H}_\omega^{\mathrm{rl}\,\mathcal{H}}$	$\mathrm{H}_\omega^{\mathrm{s}\,\mathcal{H}}$
I_1	< 0	< 0	< 0	< 0	< 0	< 0
I_2	> 0	> 0	≤ 0	≤ 0	≤ 0	≤ 0
I_3	< 0	≥ 0	≥ 0	≥ 0	≥ 0	≥ 0
Q			> 0	≤ 0		
$\mathcal{H}$					> 0	≤ 0

Enstrophy equation: terms

- evolution equation for the enstrophy

$$\frac{\mathrm{D}\,\frac{1}{2}\,\omega_i\,\omega_i}{\mathrm{D}\,t} = \overbrace{\omega_i\,\omega_j\,S_{ij}}^{\mathrm{P}_{\omega^2}} + \overbrace{\nu\,\omega_i\,\frac{\partial^2\,\omega_i}{\partial\,x_j\,\partial\,x_j}}^{\mathrm{D}_{\omega^2}}$$

$$= \underbrace{\omega_i\,\omega_j\,S_{ij}}_{\mathrm{P}_{\omega^2}} - \underbrace{\nu\,\frac{\partial\,\omega_i}{\partial\,x_j}\,\frac{\partial\,\omega_i}{\partial\,x_j}}_{\mathrm{dis}_{\omega^2}} + \underbrace{\nu\,\frac{\partial^2\,\frac{1}{2}\,\omega_i\,\omega_i}{\partial\,x_j\,\partial\,x_j}}_{\mathrm{dif}_{\omega^2}}$$

- Vortex Stretching Rate

$$\text{V.S.R.} = \frac{\omega_i\,\omega_j\,S_{ij}}{\omega_k\,\omega_k}$$

Detailed analysis for isotropic turbulence

Global criteria

	H_ω	Δ	Q	λ_2
V	42–44%	62–70%	39–42%	39–43%
k	0.96–0.98	0.98–0.99	0.96–0.98	0.97–0.99
ϵ	1.09–1.18	0.83–0.89	0.75–0.82	0.72–0.81
ω^2	1.51–1.55	1.24–1.36	1.66–1.82	1.61–1.78
P_{ω^2}	1.66-1.72	1.14–1.18	1.42–1.49	1.27–1.34
dis_{ω^2}	0.82–0.85	0.99–1.01	1.0–1.01	1.02–1.05
D_{ω^2}	1.89–2.08	1.2–1.27	1.56–1.65	1.56–1.68
V. S. R.	1.25–1.28	0.72–0.73	0.74–0.79	0.63–0.75
$\langle\Omega^2\rangle/\langle S^2\rangle$	1.28–1.42	1.39–1.57	2.02–2.31	2.0–2.36

Detailed analysis for isotropic turbulence

Subdivisions of H_ω

	$\mathrm{H}_\omega^{\mathrm{p}}$	$\mathrm{H}_\omega^{\mathrm{t}}$	$\mathrm{H}_\omega^{\mathrm{rl}\,\mathcal{H}}$	$\mathrm{H}_\omega^{\mathrm{s}\,\mathcal{H}}$
V	6–7%	11–12%	20–21%	4–5%
k	0.95–1.0	0.96–0.99	0.95–0.98	0.95–0.98
ϵ	1.09–1.39	1.11–1.33	1.04–1.07	1.07–1.1
ω^2	2.17–2.23	1.88–1.93	1.21–1.23	1.10–1.12
P_{ω^2}	2.57-2.85	2.25–2.34	1.2–1.26	1.2
dis_{ω^2}	0.90–1.05	0.87–0.92	0.77–0.84	0.67–0.72
D_{ω^2}	3.14–3.71	2.5–2.9	1.33–1.4	1.02–1.11
V. S. R.	1.37–1.51	1.38–1.43	1.09–1.19	1.16–1.18
$\langle\Omega^2\rangle/\langle S^2\rangle$	1.56–2.06	1.41–1.72	1.14–1.18	1.02–1.04