Single-particle dispersion in turbulent convection

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Lagrangian particles in turbulence

A suspension of Lagrangian fluid particles:

$$\dot{\boldsymbol{X}} = \boldsymbol{u}$$
$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \nu \Delta \boldsymbol{u}$$

One-point, two-times statistics

Lagrangian velocity increments: $\delta v_\ell(t) = \langle [\boldsymbol{v}(t) - \boldsymbol{v}(0)] \cdot \boldsymbol{\ell}/\ell \rangle$.

Two-points, one-time statistics

Pair separation: $R(t) = \langle |\boldsymbol{X}_1(t) - \boldsymbol{X}_2(t)| \rangle$.

Many-particle statistics

Triangles, thetraedra evolution.

Behaviour in hydrodynamical turbulence

If $\delta_r v \sim r^h$, with the classical Kolmogorov scaling $h = 1/3$ we expect:

Lagrangian velocity increments

$\delta v_\ell(t) \sim \delta u_E(X(t)) \sim t^{\frac{h}{1-h}}$:

$$\delta v_\ell(t) \sim t^{1/2}$$

(Obukhov, 1941)

Pair separation

$R(t) \sim t^{\frac{1}{1-h}}$:

$$R(t) \sim t^{3/2}$$

(Richardson, 1926)

2D convection in Boussinesq approximation

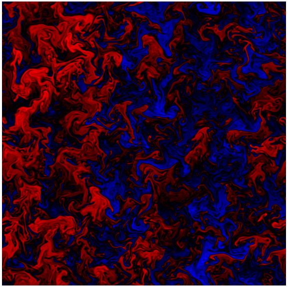
Buoyancy forces on a fluid parcel:

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \nu \Delta \boldsymbol{u} + \frac{\rho}{\rho_{av}} \boldsymbol{g}$$

$$\nabla \cdot \boldsymbol{u} = 0$$

$$\partial_t T + (\boldsymbol{u} \cdot \nabla) T = \kappa \Delta T$$

$$\rho = \rho_{av}[1 - \beta(T - T_{av})]$$



$$\partial_t \omega + (\boldsymbol{u} \cdot \nabla)\omega = \nu \Delta \omega - \beta \boldsymbol{g} \times \nabla T \qquad \nabla \times \boldsymbol{u} = \omega$$

$$\partial_t \theta + (\boldsymbol{u} \cdot \nabla)\theta = \kappa \Delta \theta \qquad T = T_0 - \Gamma z + \theta$$

Bolgiano–Obukhov theory of turbulent convection

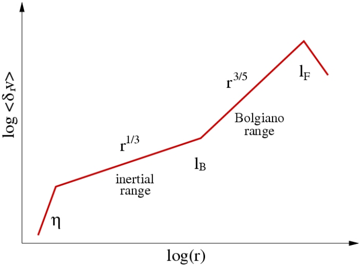
Below the Bolgiano scale ℓ_B buoyancy forces can be neglected

$$\ell_B \leftrightarrow (\boldsymbol{u} \cdot \nabla)\omega \approx \beta \boldsymbol{g} \times \nabla\theta$$
$$\ell_B = (\beta g)^{-3/2} \epsilon_u^{5/4} \epsilon_\theta^{-3/4}$$

How do velocity increments scale in the Bolgiano range ($\ell_B < r < l_F$)?

$$\begin{cases} \delta_r u^2 r^{-2} \approx \beta g \delta_r \theta r^{-1} \\ \delta_r \theta^2 r^{-1} \delta_r u \approx \epsilon_\theta \end{cases}$$

$$\begin{cases} \delta_r u \approx (\beta g)^{2/5} \epsilon_\theta^{1/5} r^{3/5} \\ \delta_r \theta \approx (\beta g)^{-1/5} \epsilon_\theta^{2/5} r^{1/5} \end{cases}$$



In K41 $\quad h = 1/3$

$$\Rightarrow E(k) \sim k^{-5/3}$$

In BO $\quad h = 3/5$

$$\Rightarrow E(k) \sim k^{-11/5}$$

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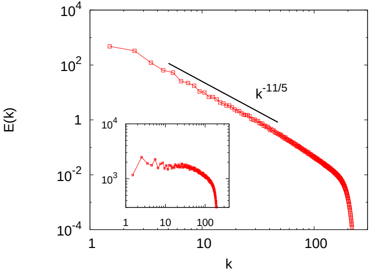

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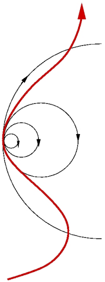
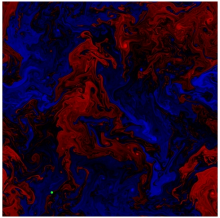
Turbulent one-particle dispersion I

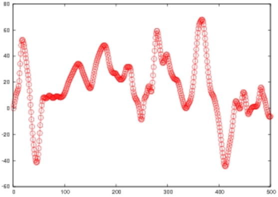
Lagrangian velocity increments

$$\delta v_\ell(t) = \langle [\boldsymbol{v}(t) - \boldsymbol{v}(t_0)] \cdot \boldsymbol{\ell}/\ell \rangle$$

along a Lagrangian trajectory. At every t we can see $\boldsymbol{v}(t)$ as the result of contributions coming from all eddies.







The typical (turnover) time of an eddy of size r is:

$$\tau(r) \approx \frac{r}{\delta_E u} \sim r^{1-h}$$

if the Eulerian velocity increments scale as r^h.

Turbulent one-particle dispersion II

Small eddies ($t \gg \tau$)

No correlation

Medium eddies ($t \approx \tau$)

Large eddies ($t \ll \tau$)

Turbulent one-particle dispersion II

Small eddies ($t \gg \tau$)

No correlation

Medium eddies ($t \approx \tau$)

Their size scales as $r \sim t^{\frac{1}{1-h}}$
Their contribution is

$$\delta v \sim \delta_E v(r(t)) \sim r(t)^h \sim t^{\frac{h}{1-h}}$$

Large eddies ($t \ll \tau$)

Turbulent one-particle dispersion II

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Large eddies ($t \ll \tau$)

$$\delta v = \frac{\partial v(t_0)}{\partial t} t + \mathcal{O}(t^2)$$

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No correlation

Medium eddies ($t \approx \tau$)

Their size scales as $r \sim t^{\frac{1}{1-h}}$

Their contribution is

$$\delta v \sim \delta_E v(r(t)) \sim r(t)^h \sim t^{\frac{h}{1-h}}$$

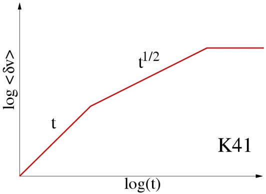
Large eddies ($t \ll \tau$)

$$\delta v = \frac{\partial v(t_0)}{\partial t} t + \mathcal{O}(t^2)$$

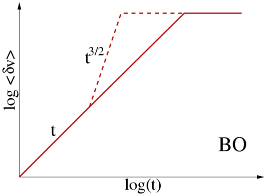
$$\delta v \simeq \partial_t v(t_0)\, t + c(t_0)\, t^{\frac{h}{1-h}}$$

$$\Rightarrow \delta v \sim t^q \quad q = \min(1, \frac{h}{1-h})$$

K41:
$h = 1/3$
$\delta v \sim t^{1/2}$

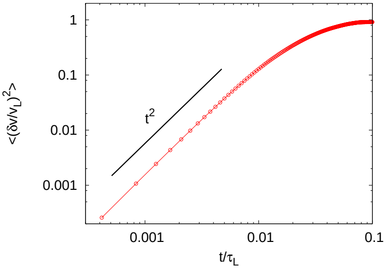


BO:
$h = 3/5$
$\delta v \sim t$



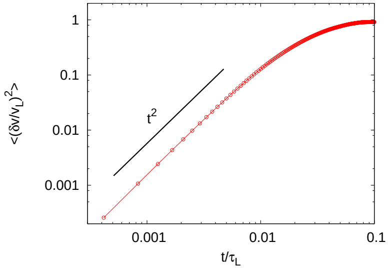
Are inertial-range contributions invisible?

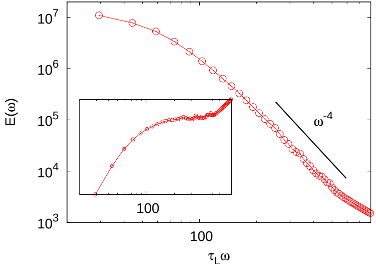
Check: δv^2 indeed scales as t^2.



Are inertial-range contributions invisible?

Check: δv^2 indeed scales as t^2. Should we only care about large scales? How to treat *more than smooth* signals?





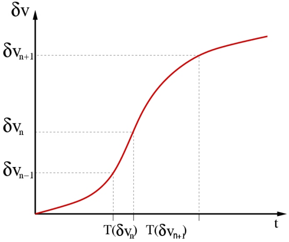
The spectrum of a smooth signal decays at least exponentially, while the $t^{3/2}$ component gives an observed ω^{-4} tail.

Inverse statistics

Exit-time $T_\rho(\delta_v)$: time for velocity fluctuations to grow from δv_n to δv_{n+1}.

$$\delta v_n = \rho^n \delta v_0 \quad (\rho > 1)$$

Analyzing $T_\rho(\delta v)$ we remove crossover effects due to scale mixing.

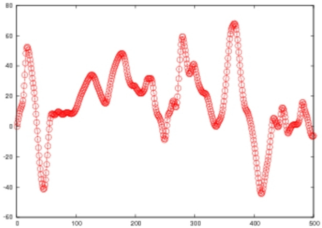


Convective velocity fluctuations: $\delta v \sim t^{3/2} \longrightarrow T(\delta v) \sim \delta v^{2/3}$

All samples in $T(\delta v)$ belong to the same scale $\Rightarrow$ we can separate contributions

Moments of exit-times for velocity increments I

$$\delta v = \partial_t v(t_0)\, t + c(t_0) t^q \; ; \quad q = 3/2$$



The scaling of exit-times moments?

- $T^p(\delta v) \sim \delta v^p$ almost everywhere
- $T^p(\delta v) \sim \delta v^{p/q}$ where $\partial_t v = 0$

For $1 \leq q < 2$ the first derivative of the signal is self-affine ($0 \leq q - 1 < 1$), vanishing on a fractal set of dimension

$$D = 1 - (q - 1)$$

The probability of $\partial_t v = 0$ is thus equal to the probability of picking a point at random on a fractal set of dimension D:

$$P(T \sim \delta v^{1/q}) \sim T^{1-D} = \delta v^{1-1/q}$$

Moments of exit-times for velocity increments II

$$\begin{cases} T^p(\delta v) \sim \delta v^p + P(T \sim \delta v^{1/q}) \cdot \delta v^{p/q} \\ P(T \sim \delta v^{1/q}) \sim \delta v^{1-1/q} \end{cases}$$

$$T^p(\delta v) \sim \delta v^{\chi(p)}, \qquad \chi(p) = \min(p, \frac{p}{q} + 1 - \frac{1}{q}), \qquad q = \frac{3}{2}$$

Moments of exit-times for velocity increments II

$$\begin{cases} T^p(\delta v) \sim \delta v^p + P(T \sim \delta v^{1/q}) \cdot \delta v^{p/q} \\ P(T \sim \delta v^{1/q}) \sim \delta v^{1-1/q} \end{cases}$$

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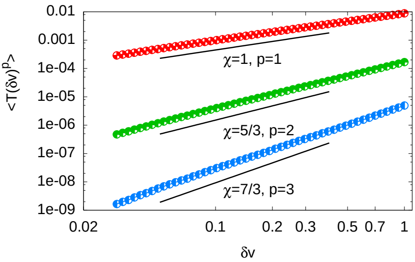
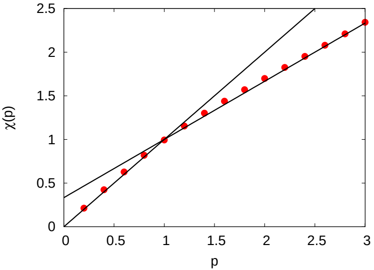
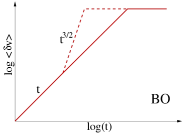
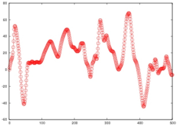




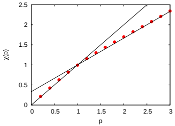
A physically relevant model displaying non pseudodiffusive Lagrangian velocity increments



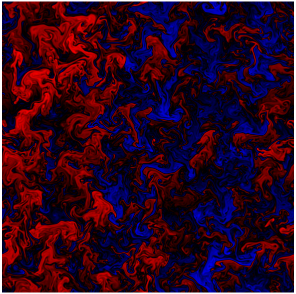
A difference in predictions for scale-dependent and scale-independent observables



A nontrivial prediction on moments of velocity increments



Direct Numerical Simulations



- Resolution: 1024^2, 2048^2
- 64000 particles
- Over 150 runs
- Pseudospectral code employing hyperviscosity, hyperdiffusivity, friction
- Periodic boundary conditions

Simulations ran on Turbofarm^2 cluster in Torino

Turbulent two-particle dispersion I

Pair separation

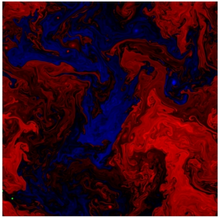
$$R(t) = \langle |\boldsymbol{x}_1(t) - \boldsymbol{x}_2(t)| \rangle$$

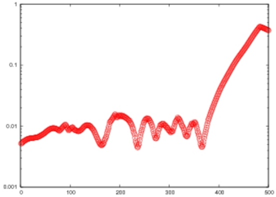
We expect two-particle dispersion to be selfsimilar, just like the velocity field.
If $\delta_r v \sim r^h$:

$$\delta v \sim \frac{r}{t} \sim r^h \quad \Rightarrow r \sim t^{\frac{1}{1-h}}$$

So separation depends on h:

- in K41 $h = 1/3$ and $R(t) \sim t^{3/2}$ (Richardson, 1926)
- in BO $h = 3/5$ and $R(t) \sim t^{5/2}$





Turbulent 2-particle dispersion II

Indeed, the moment of order 2 scales as t^5

$$R(t) \sim t^{5/2}$$

$$\Rightarrow R^2(t) \sim t^5$$