

Magnetoelliptic Instability in the Presence of Inertial Forces

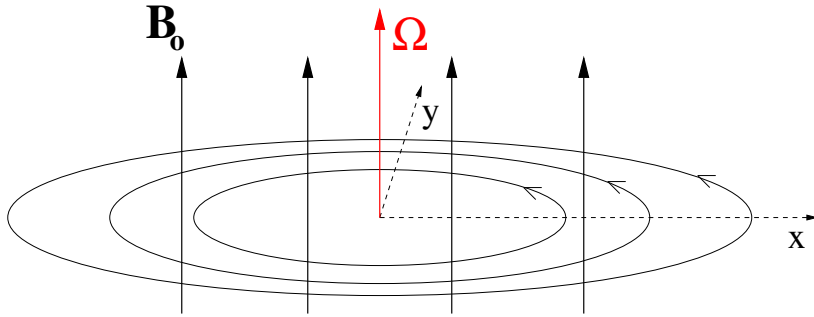


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$$\mathbf{u}_0 = \omega \left[-Ey, E^{-1}x, 0 \right] = \gamma \left[-(1 + \epsilon) y, (1 - \epsilon) x, 0 \right],$$

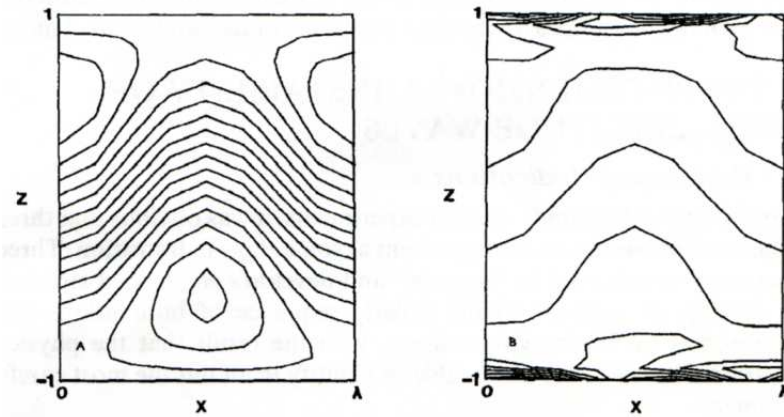
Parameters:



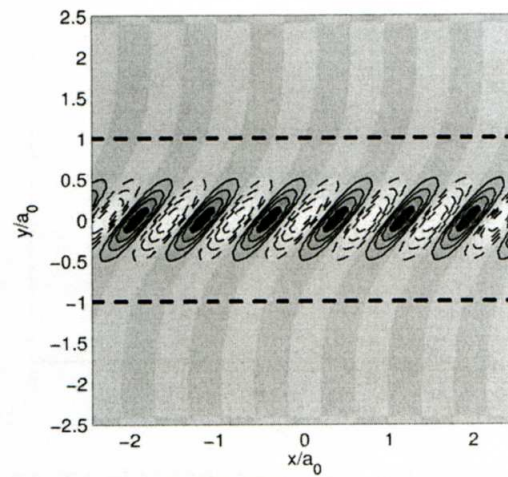
$$\left. \begin{aligned} E \\ \epsilon &= \frac{E^2 - 1}{E^2 + 1} \\ \zeta &= \frac{1}{2} (E - E^{-1}) \\ \delta &= \frac{1}{2} (E + E^{-1}) \end{aligned} \right\} \text{ - ellipticity}$$

$$\left. \begin{aligned} \mathcal{R}_v &\doteq Ro^{-1} \\ \mathcal{R}_i &\doteq \mathcal{R}_v \delta \end{aligned} \right\} \text{ - Coriolis force strength}$$

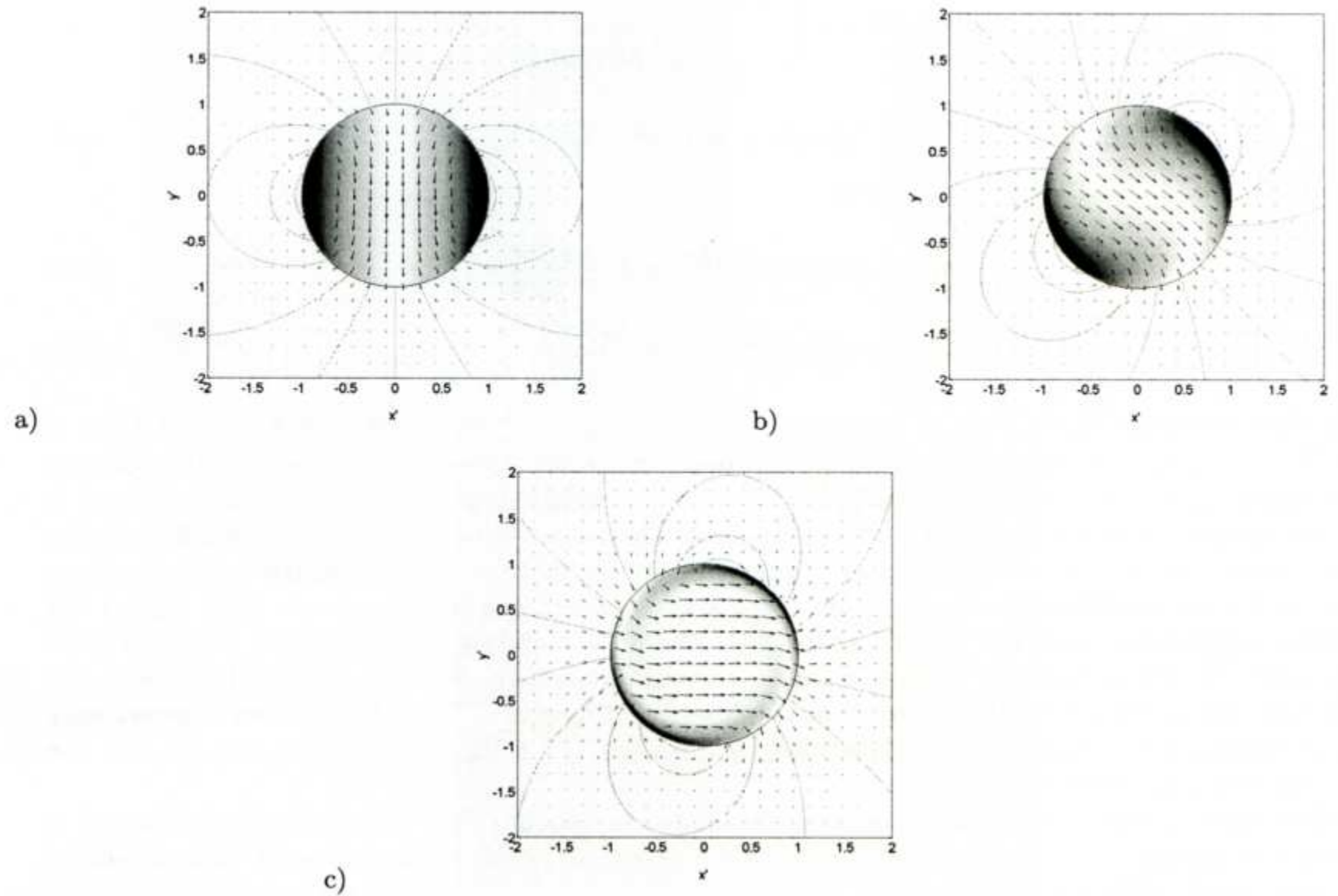
$$\left. \begin{aligned} h &= \frac{k_0 B_0}{\sqrt{\mu_0 \rho} \gamma} \\ H &= h \delta \end{aligned} \right\} \text{ - Magnetic field strength}$$



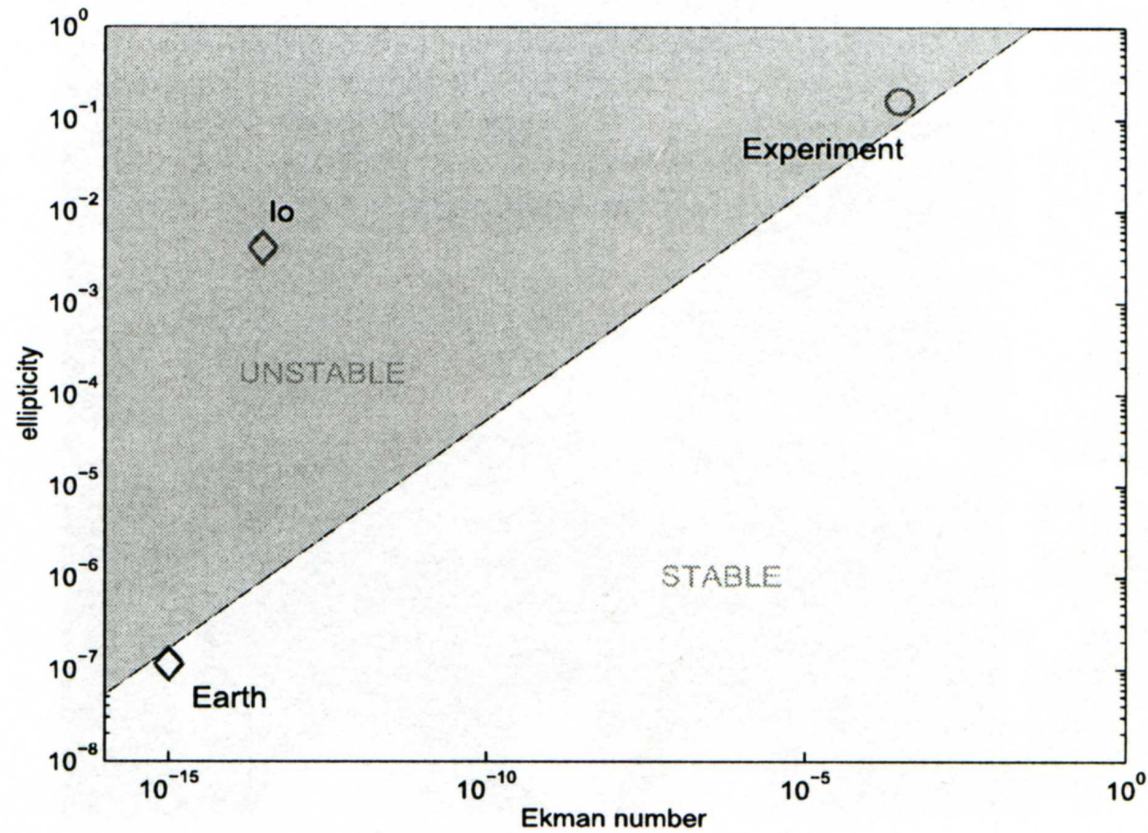
Streamlines (A) and vorticity lines (B) in Poisseuille flow for $Re = 4000$ (after Orszag & Patera 1981).



Elliptical structures in strained shear layer (after Le Dizes 2003).



Magnetic field experimentally induced by elliptical instability. Greyscale indicates Joule dissipation (proportional to $|\nabla \times \mathbf{H}|^2$) in the interior of the spheroid. (a) $Re_m = 0.01$, (b) $Re_m = 10$, (c) $Re_m = 100$ (after Lacaze *et al.* 2006).



Stability of a flow in an ellipsoid depended on the Ekman number and ellipticity (after Lacaze *et al.* 2006).

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla p - 2\boldsymbol{\Omega} \times \mathbf{u} + \frac{1}{\rho} \mathbf{j} \times \mathbf{B}, \quad (\text{Euler equation})$$

$$\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{B} = (\mathbf{B} \cdot \nabla) \mathbf{u}, \quad (\text{Induction equation})$$

$$\nabla \cdot \mathbf{u} = 0, \quad \nabla \cdot \mathbf{B} = 0, \quad (\text{mass conservation and the Gauss law})$$

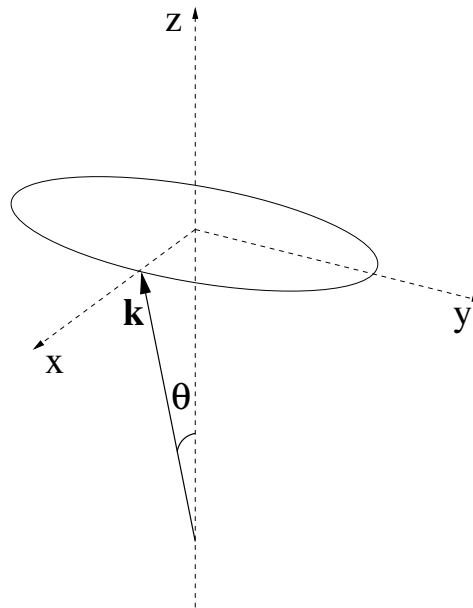
$$\mathbf{j}(\mathbf{x}, t) = \frac{1}{\mu_0} \nabla \times \mathbf{B}(\mathbf{x}, t). \quad (\text{Maxwell law})$$

Perturbations of the form (*linear stability*):

$$\begin{pmatrix} \mathbf{u}' \\ \mathbf{B}' \\ p' \end{pmatrix} = \begin{pmatrix} \mathbf{v}(t) \\ \mathbf{b}(t) \\ \tilde{p}(t) \end{pmatrix} \exp [i\mathbf{k}(t) \cdot \mathbf{x}]$$

$$\frac{d\mathbf{s}}{d\tau} = \widehat{\mathcal{S}}(\tau) \mathbf{s}$$

$$\tau = \omega (t - t_0), \quad \mathbf{s} = \begin{bmatrix} v_x \\ v_y \\ b_x \\ b_y \end{bmatrix}$$



End of wave vector \mathbf{k} moves along an ellipse.

and

$$\widehat{\mathcal{S}}(\tau) = \begin{bmatrix} \frac{2k_x k_y}{k^2} (E^{-1} + \mathcal{R}_i) & \left(1 - \frac{2k_x^2}{k^2}\right) (E + \mathcal{R}_i) + \mathcal{R}_i & iH \cos \vartheta & 0 \\ \left(\frac{2k_y^2}{k^2} - 1\right) (E^{-1} + \mathcal{R}_i) - \mathcal{R}_i & -\frac{2k_x k_y}{k^2} (E + \mathcal{R}_i) & 0 & iH \cos \vartheta \\ iH \cos \vartheta & 0 & 0 & -E \\ 0 & iH \cos \vartheta & E^{-1} & 0 \end{bmatrix}$$

Destabilized ellipticities δ :

$$\varpi_1 = -\varpi_2 = n\omega \quad \text{and} \quad n \in \mathbb{N} \Rightarrow \delta = \sqrt{\frac{n^2 - 1}{c^2 - 1}} \quad \text{i} \quad n \geq |c| \quad \text{gdzie} \quad |c| > 1,$$

$$\varpi_3 = -\varpi_4 = n\omega \quad \text{and} \quad n \in \mathbb{N} \Rightarrow \delta = \sqrt{\frac{n^2 - 1}{d^2 - 1}} \quad \text{i} \quad n \geq |d|$$

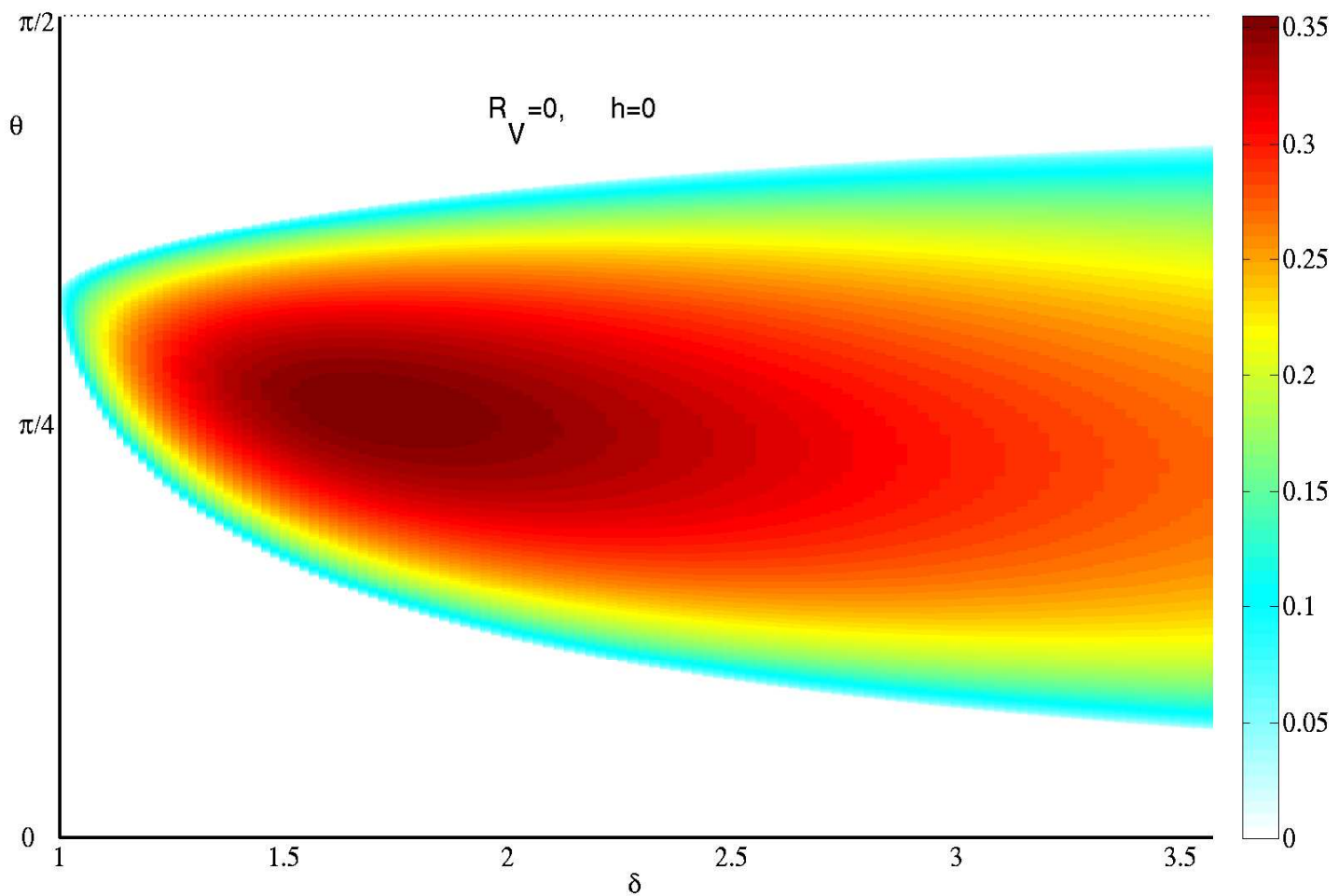
Destabilized directions of propagation ϑ :

$$\cos \vartheta = \frac{1}{(1 + \mathcal{R}_i) + \sqrt{(1 + \mathcal{R}_i)^2 + H^2}} \quad \text{and} \quad \sigma_{max} = \frac{1}{4}\zeta (1 + \cos \vartheta)^2$$

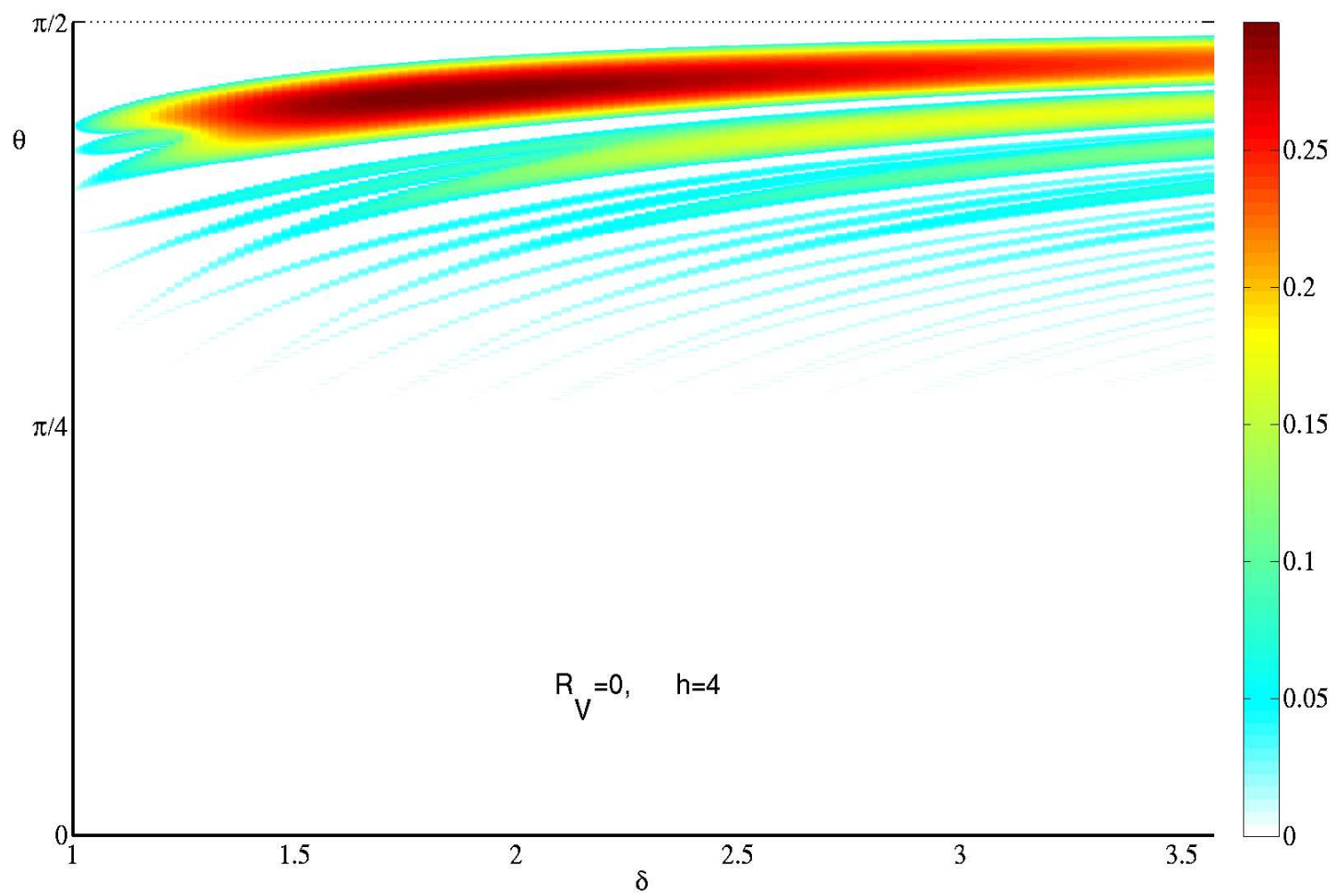
$$\cos \vartheta = \frac{1}{\sqrt{(1 + \mathcal{R}_i)^2 + H^2}} \quad \text{and} \quad \sigma_{max} = \frac{1}{4}\zeta (1 - \cos^2 \vartheta) \left(1 - (1 + \mathcal{R}_i)^2 \cos^2 \vartheta\right)$$

$$\cos \vartheta = \frac{1}{\sqrt{(1 + \mathcal{R}_i)^2 + H^2} - (1 + \mathcal{R}_i)} \quad \text{and} \quad \sigma_{max} = \frac{1}{4}\zeta (1 - \cos \vartheta)^2$$

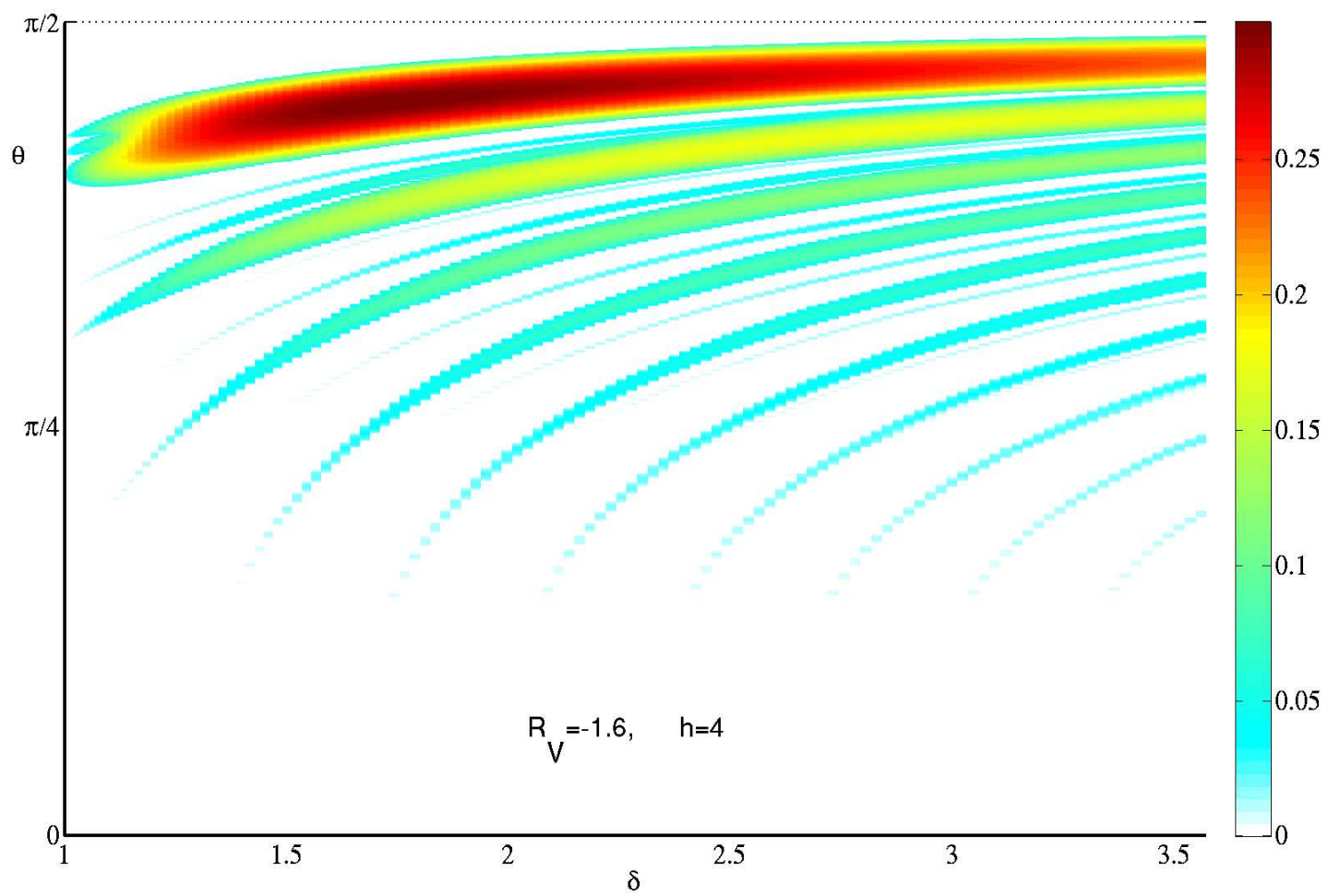
$$H \rightarrow \infty \quad \Rightarrow \quad \sigma_{max} \rightarrow \frac{1}{4}\zeta \quad \text{and} \quad \vartheta \rightarrow \frac{\pi}{2}$$



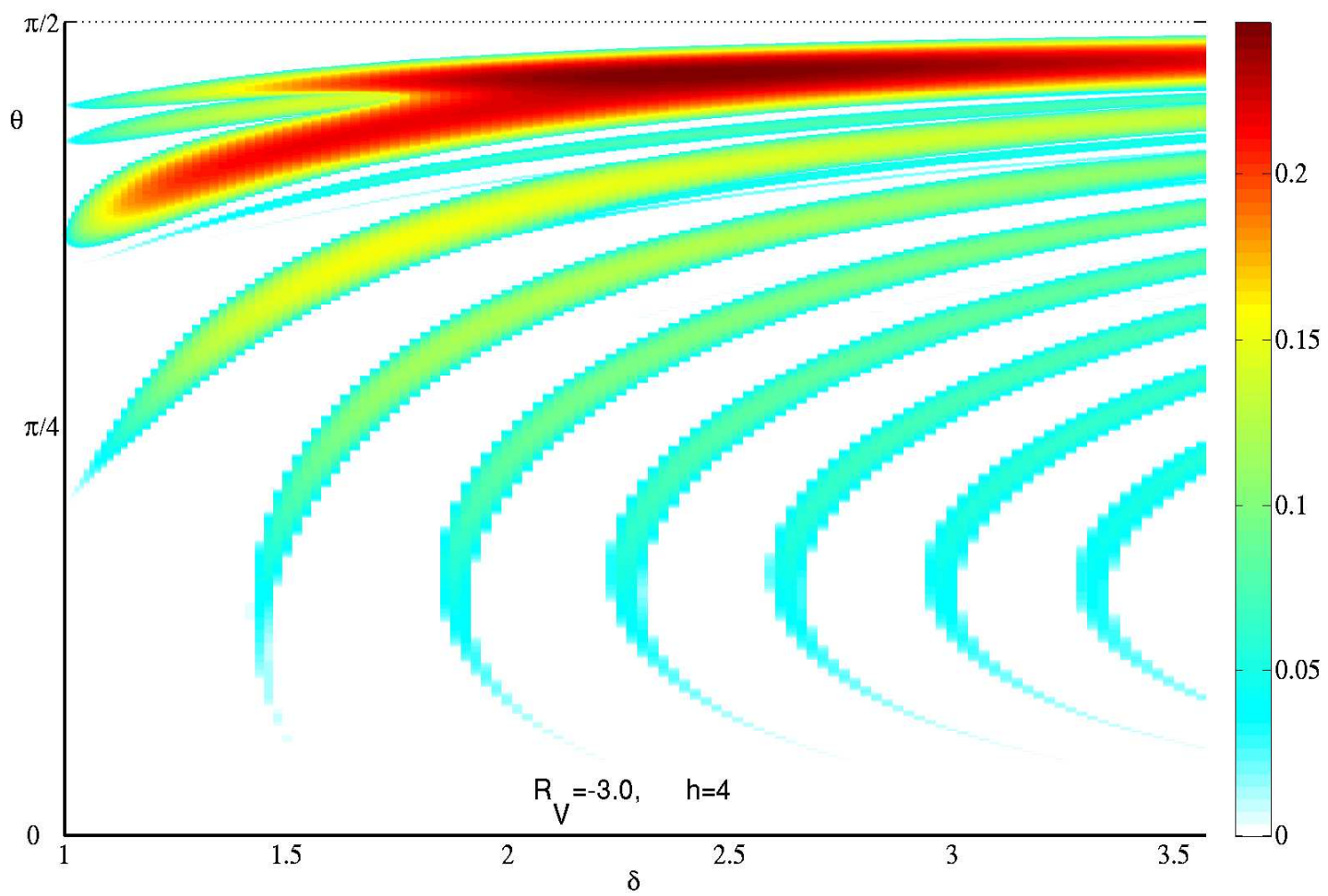
Growth rates as color-map on the δ – ϑ plane, for pure elliptical instability ($h = 0, \mathcal{R}_v = 0$).



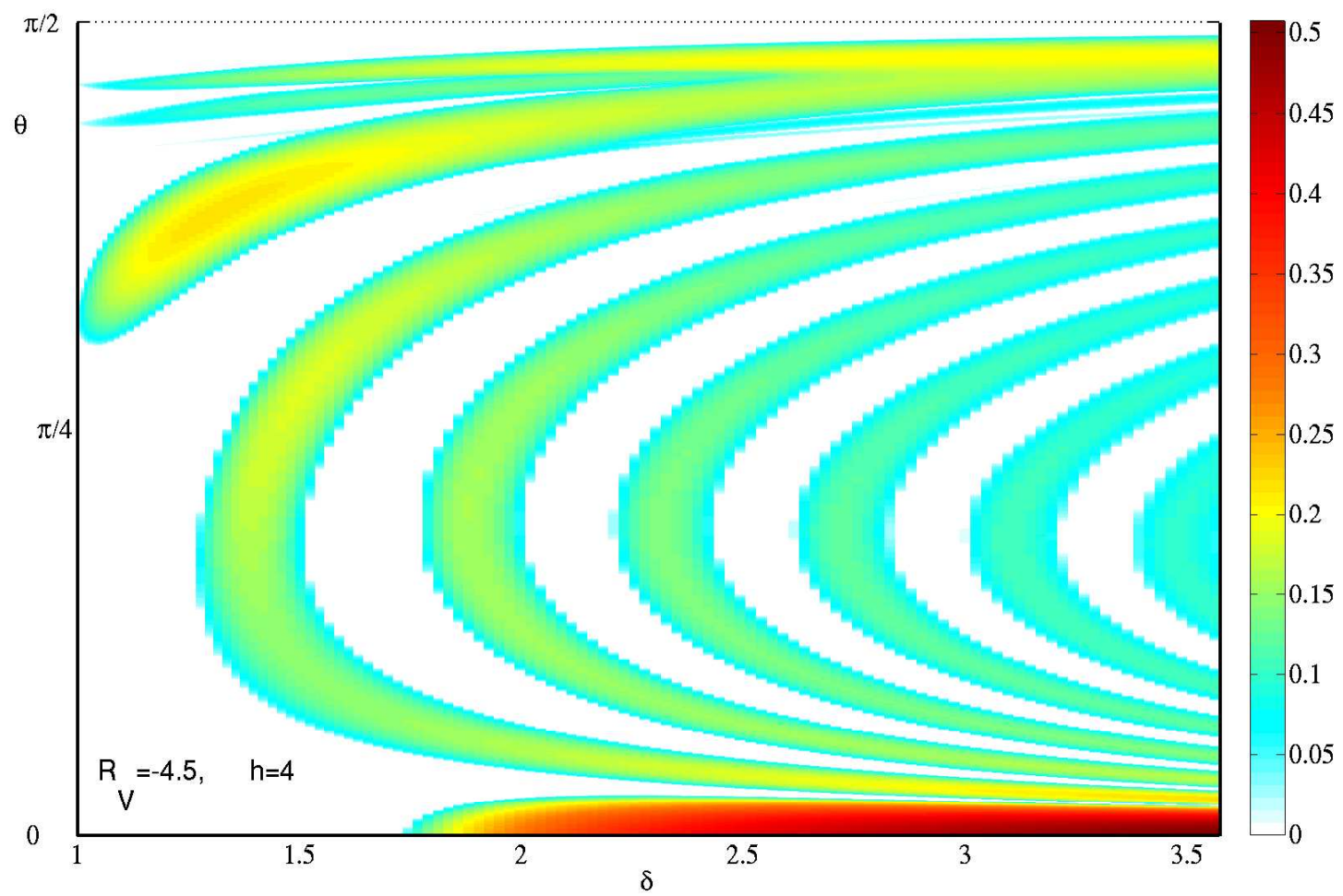
$$(h = 4, \mathcal{R}_v = 0).$$



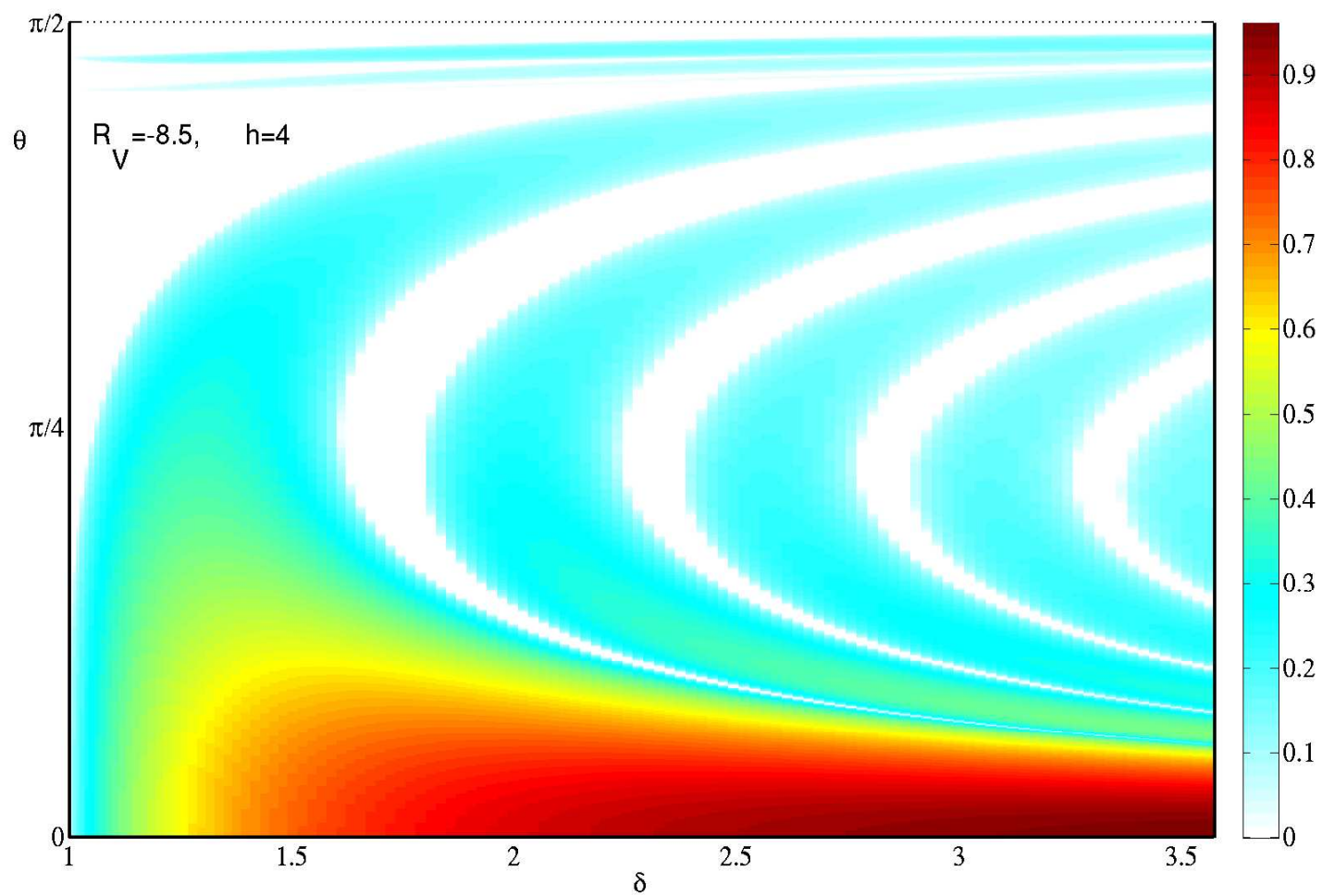
$h = 4$ and $\mathcal{R}_v = -1.6$.



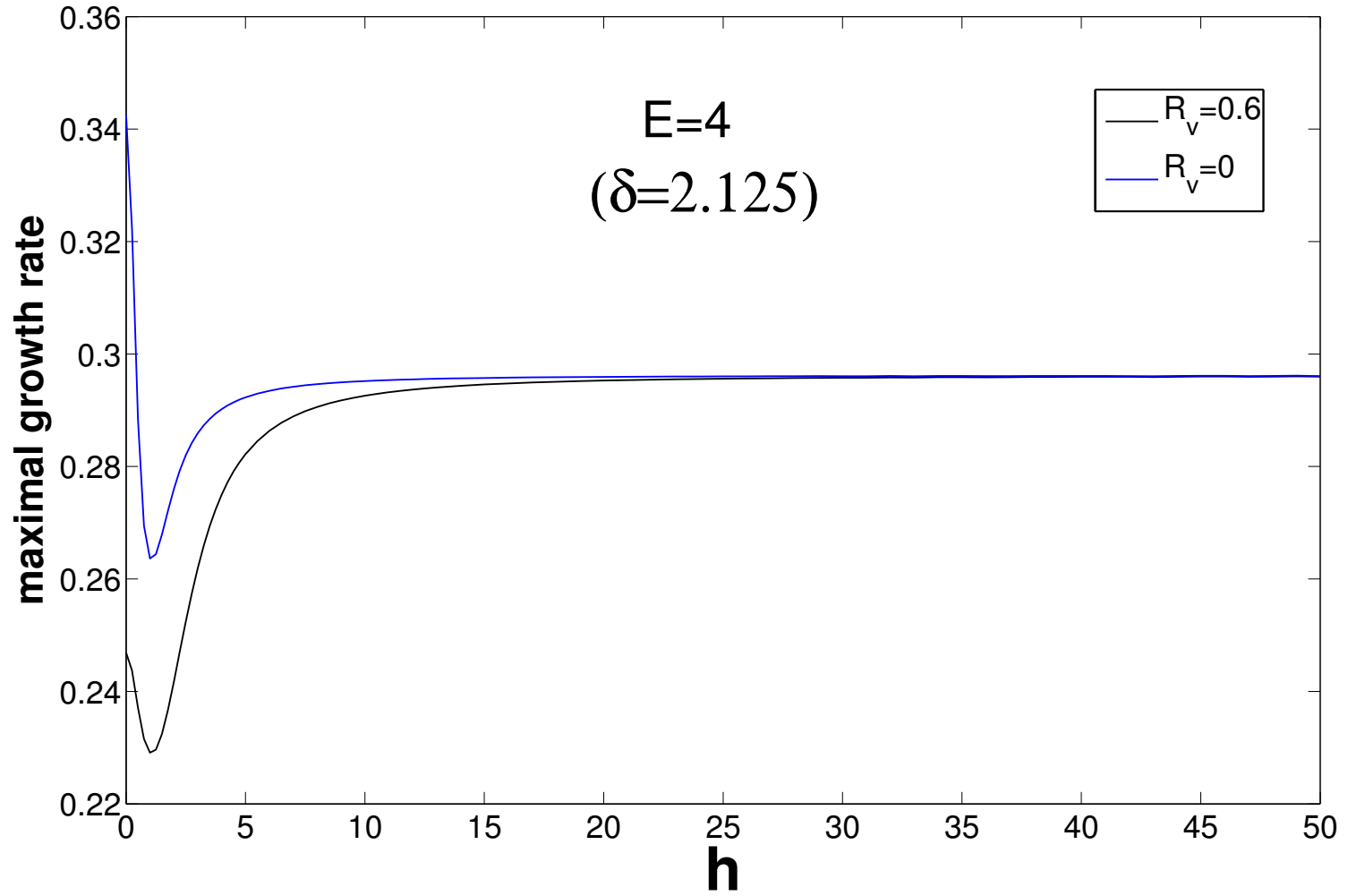
$h = 4$ and $\mathcal{R}_v = -3.0$.



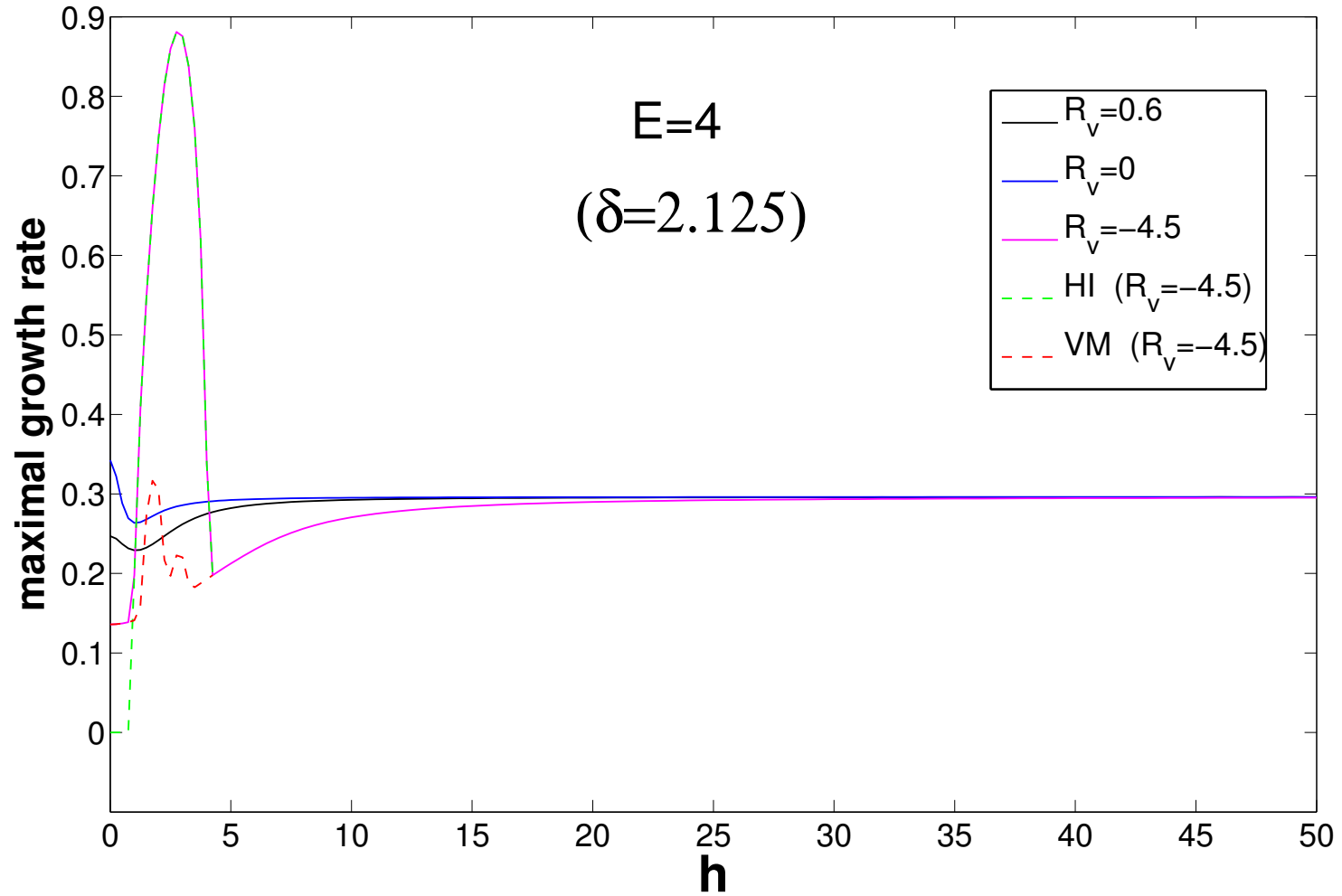
$$h = 4 \text{ and } \mathcal{R}_v = -4.5.$$



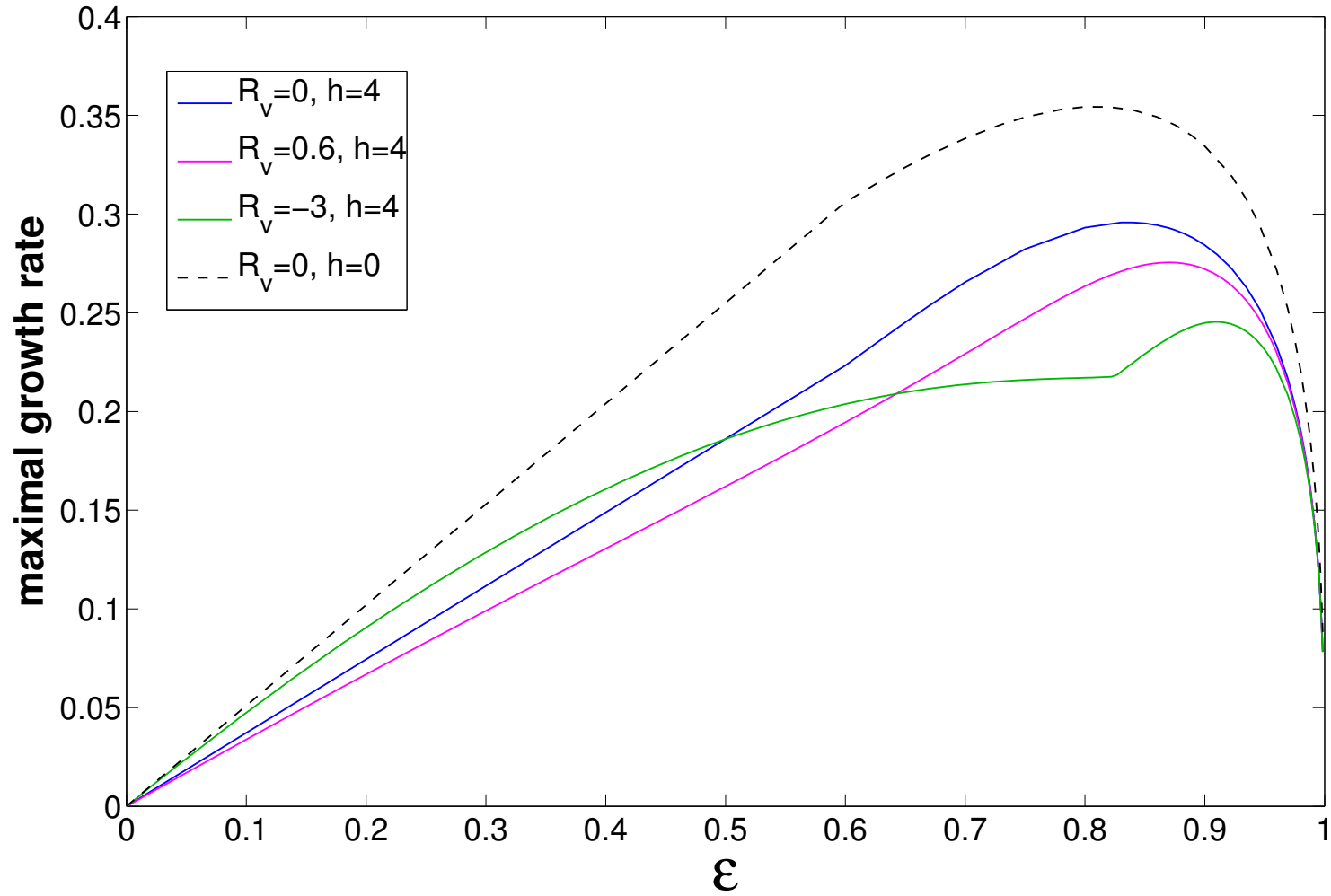
$$h = 4 \text{ and } \mathcal{R}_v = -8.5.$$



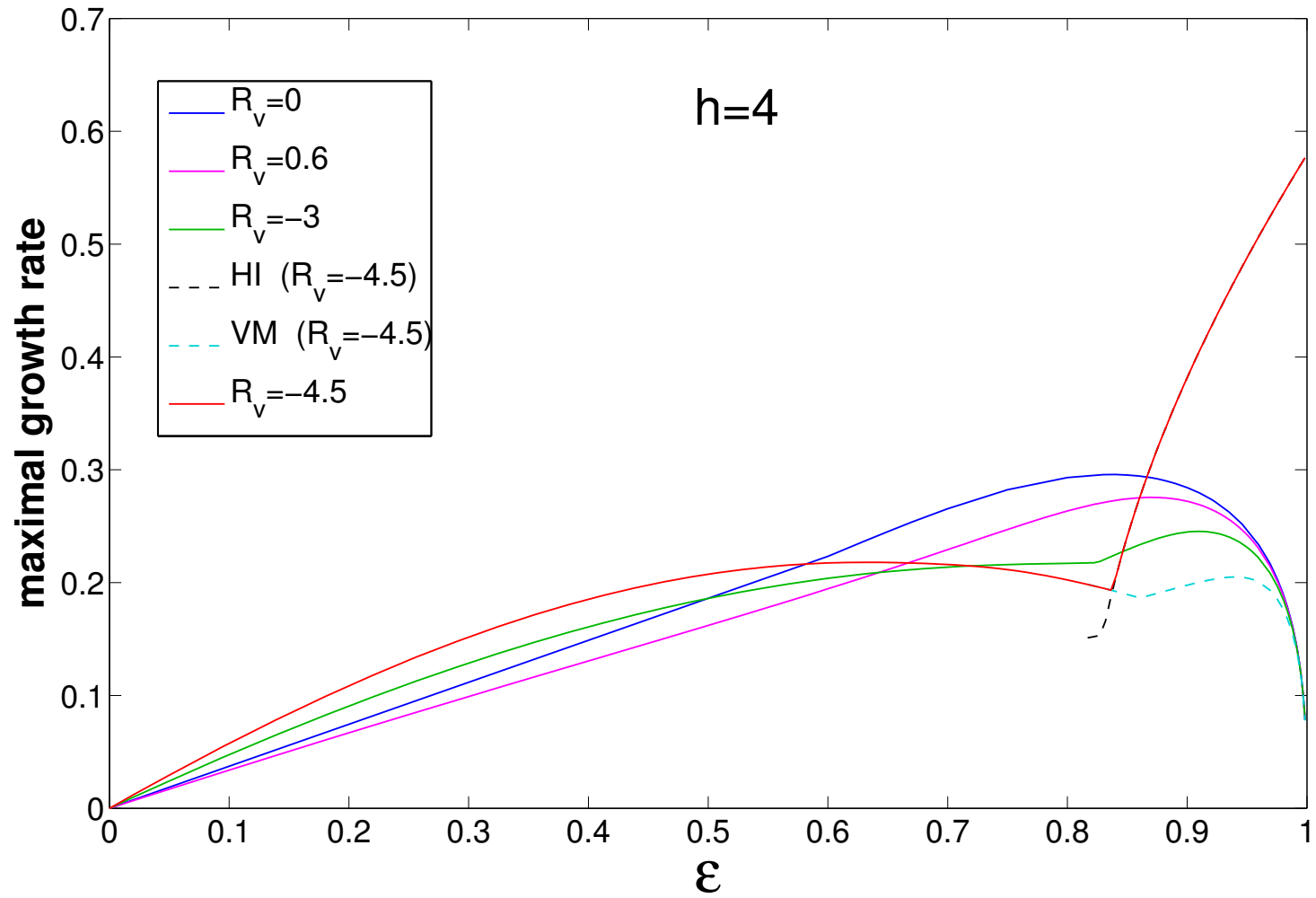
The maximal growth rate as a function of magnetic field h for eccentricity $E = 4$ and $\mathcal{R}_v = 0.6, 0$.



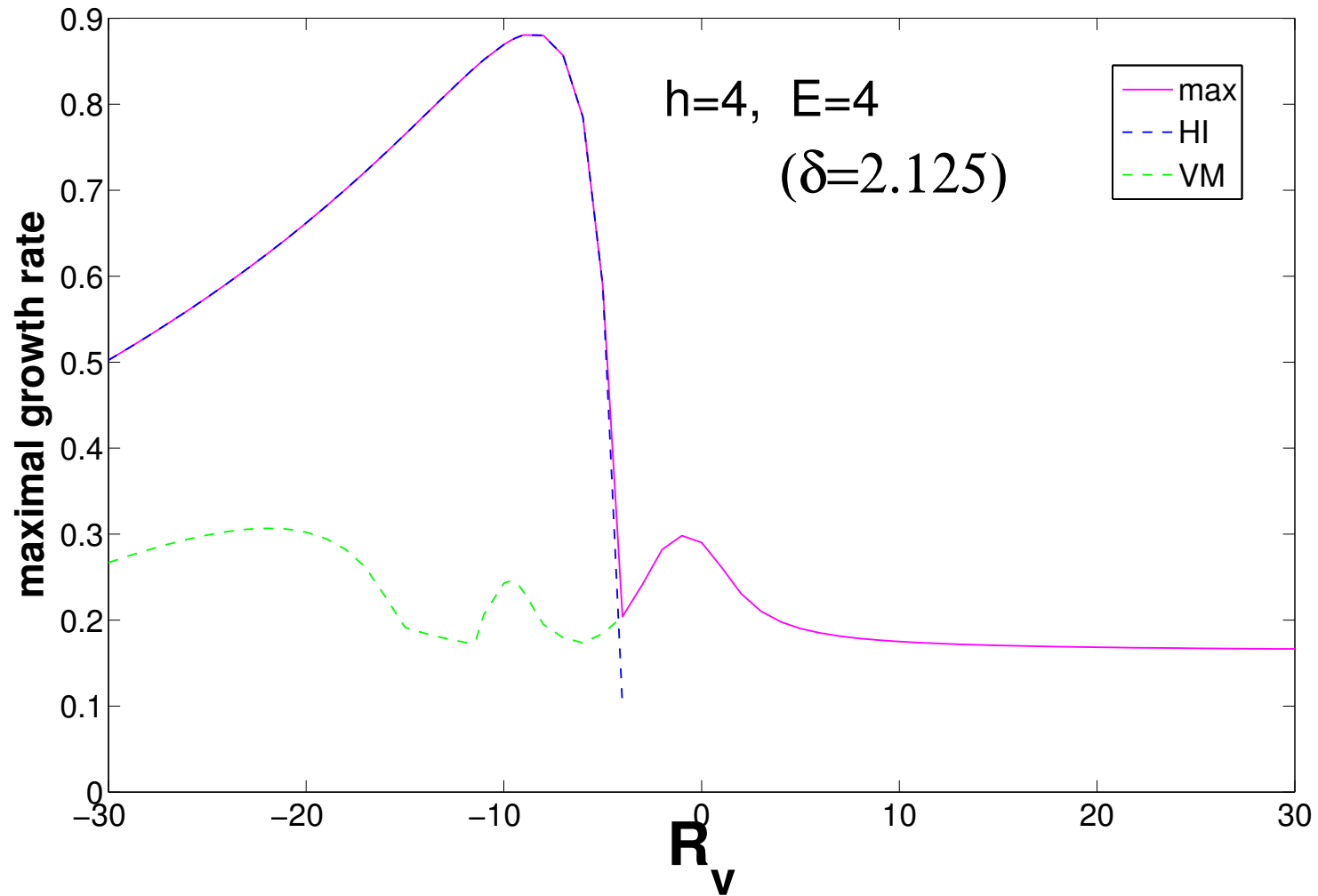
The maximal growth rate as a function of magnetic field h for eccentricity $E = 4$ and $\mathcal{R}_v = 0.6, 0$ i $- 4.5$. VM stands for *vertical mode* and HI for *horizontal instability*.



The maximal growth rate as a function of ellipticity ϵ for $h = 4$ and $\mathcal{R}_v = 0, 0.6, 3.0$ (solid lines), and for $h = 0$ i $\mathcal{R}_v = 0$ (dashed line).



The maximal growth rate as a function of ellipticity ϵ for $h = 4$ and $\mathcal{R}_v = 0, 0.6, 3.0i - 4.5$.



The maximal growth rate as a function of parameter measuring the Coriolis force strength \mathcal{R}_v for $h = 4$ i $E = 4$.

SUMMARY

- The joint influence of the Lorenz and Coriolis forces was investigated.
- Mechanism of the instability.
- Analytical formula was obtained for the destabilized directions of propagation of the perturbations and ellipticities of the basic flow.
- In strong field limit $H \rightarrow \infty$ instability persists.
- *Horizontal instability* (\mathbf{v} and $\mathbf{b} \perp OZ$), present only for anticyclonic rotation, dominates other modes, is present in Couette flow limit and exhibits singular behaviour as $H \rightarrow 0$.

CONCLUSIONS (1)

- When horizontal instability is not present both factors have *stabilizing effect* by reducing σ and the number of destabilizing directions of propagation (though the number of destabilizing rezonances increases).
- Destabilizing effect is evident when HI is present. Then, σ and the number of destabilizing directions of propagation increase. The domain in which HI is present is much larger in the presence of magnetic field.

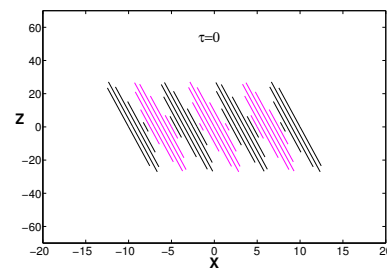
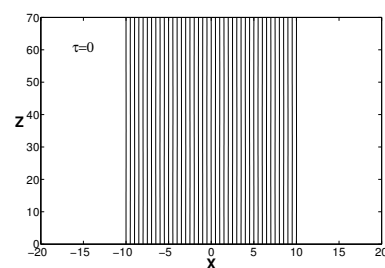
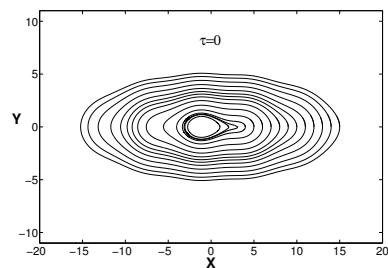
CONCLUSIONS (2)

- Upper bound for magnetic field strength in *accretion disk* ($u_A = B_0 / \sqrt{\mu_0 \rho}$ is the Alfvén velocity):

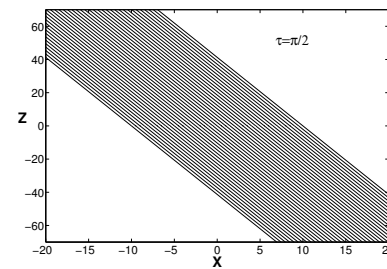
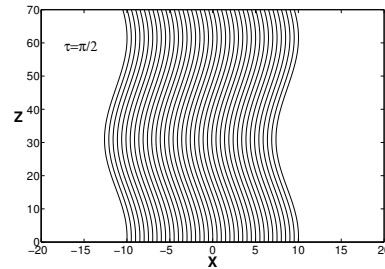
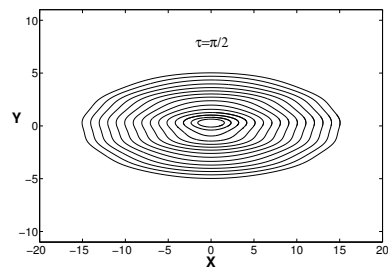
$$u_A \leq \sqrt{3 + 2\mathcal{R}_i} \frac{\omega d}{\chi} \quad \text{for } \mathcal{R}_i \geq -1$$

$$u_A \leq \sqrt{-2\mathcal{R}_i - 1} \frac{\omega d}{\chi} \quad \text{for } \mathcal{R}_i < -1$$

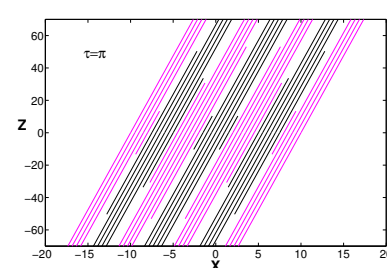
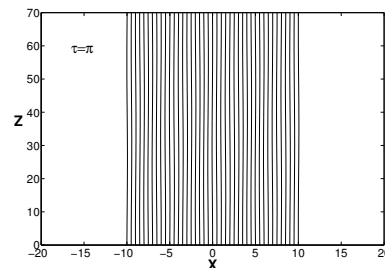
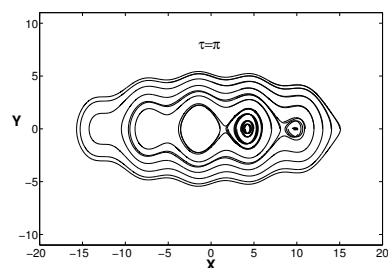
- The possibility of damping the instability by dissipation is smallest in a system with anticyclonic rotation, with HI present ($\mathcal{R}_v < -\frac{h^2}{4}$), therefore in such systems turbulence is likely to appear.



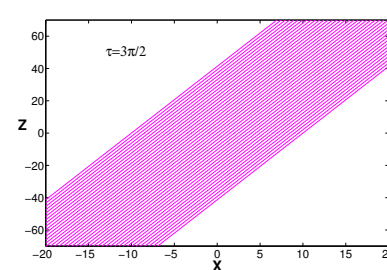
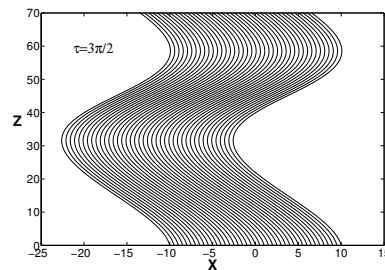
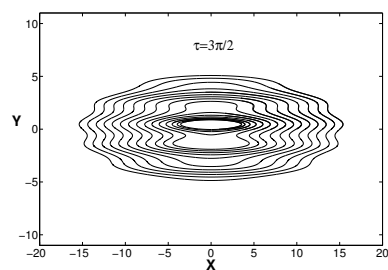
$$\tau = 0$$



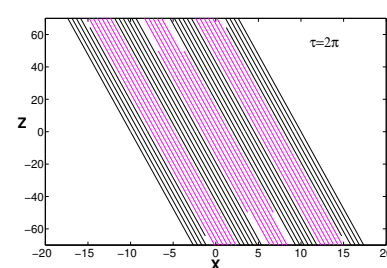
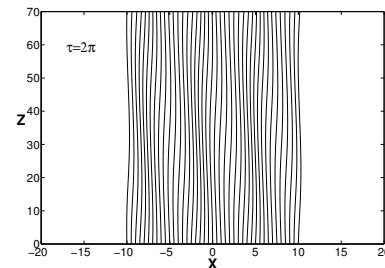
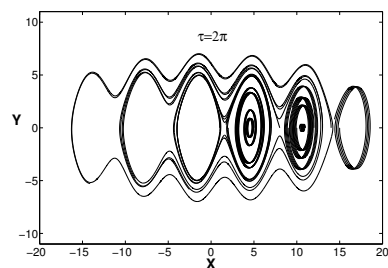
$$\tau = \pi/2$$



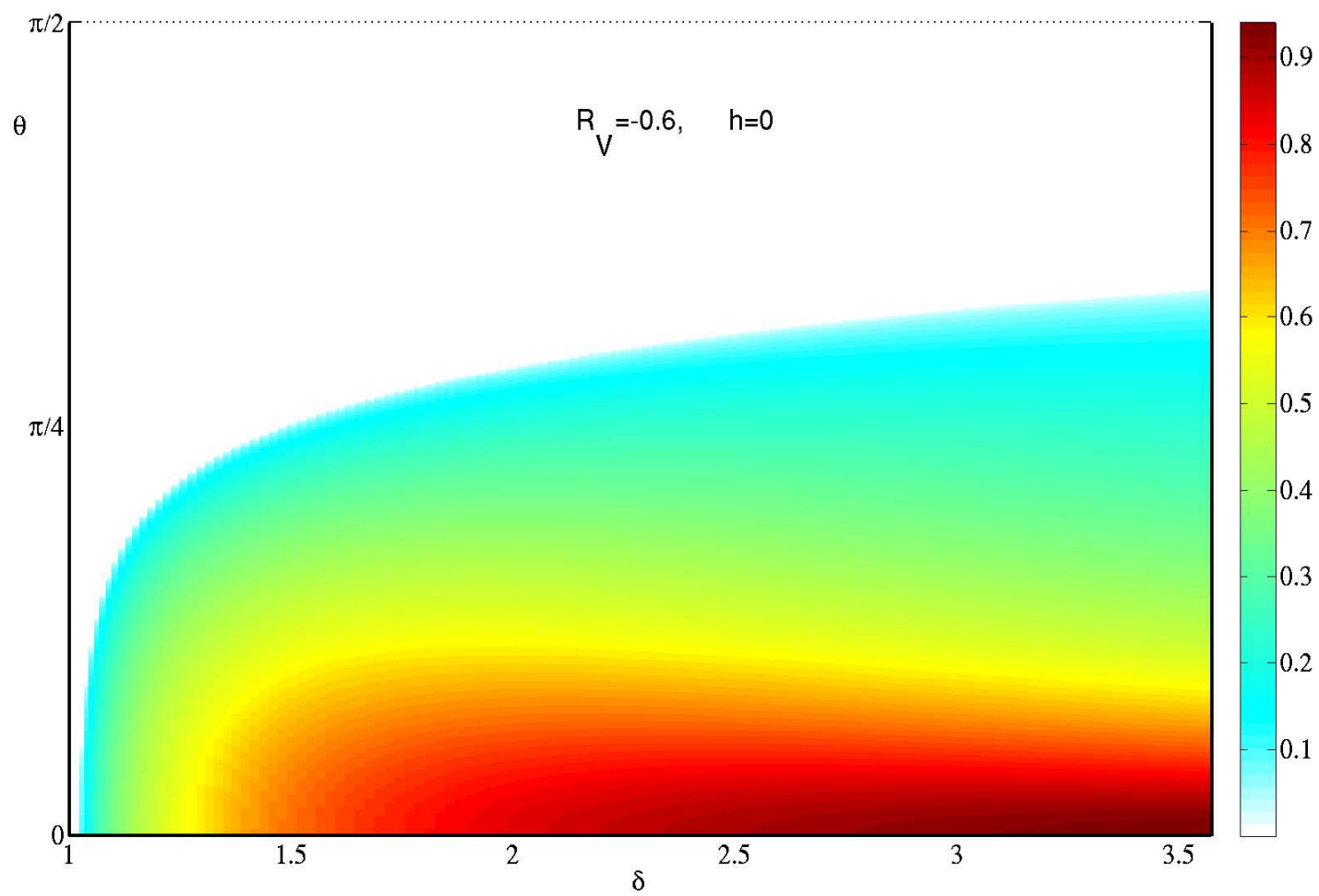
$$\tau = \pi$$



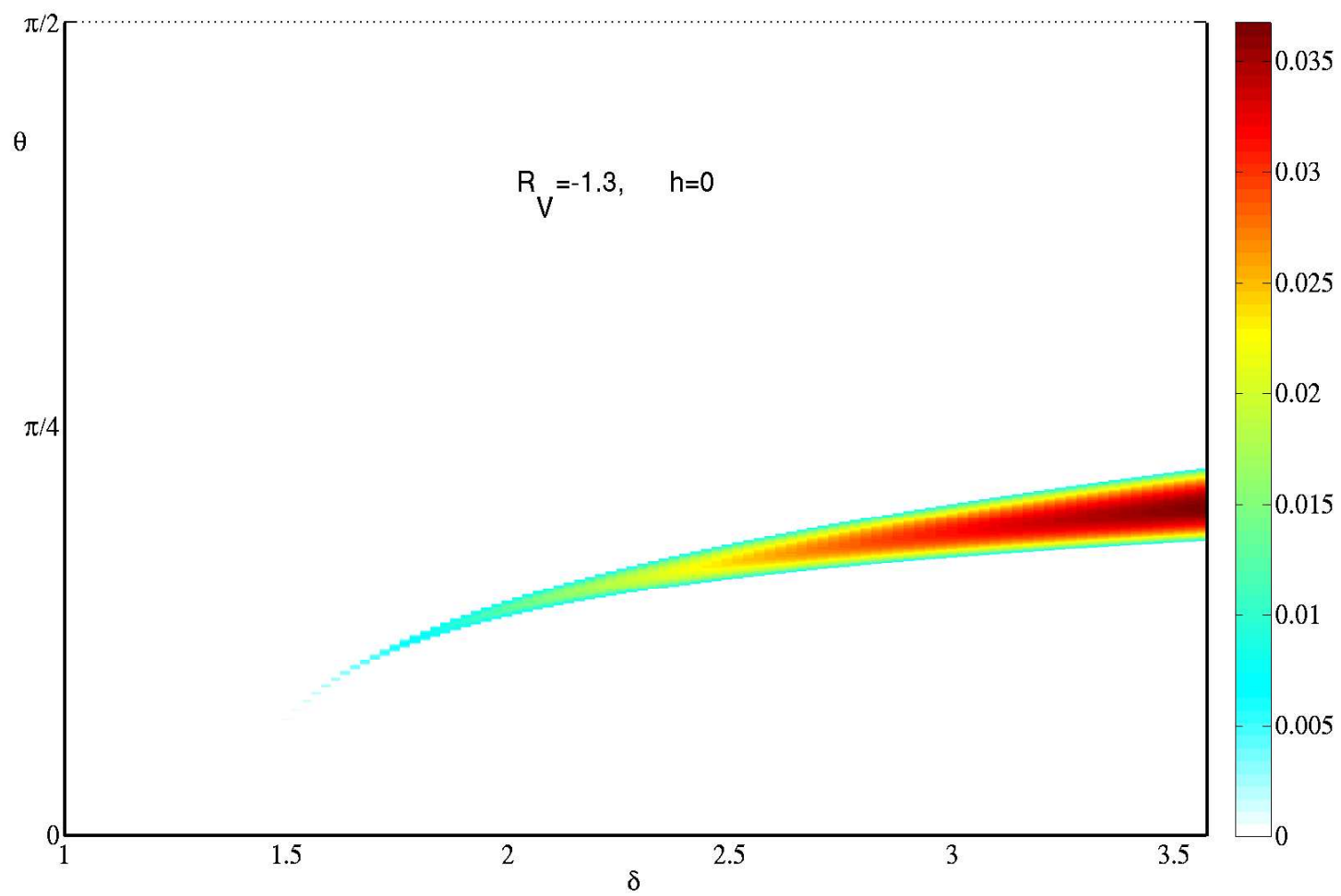
$$\tau = 3\pi/2$$



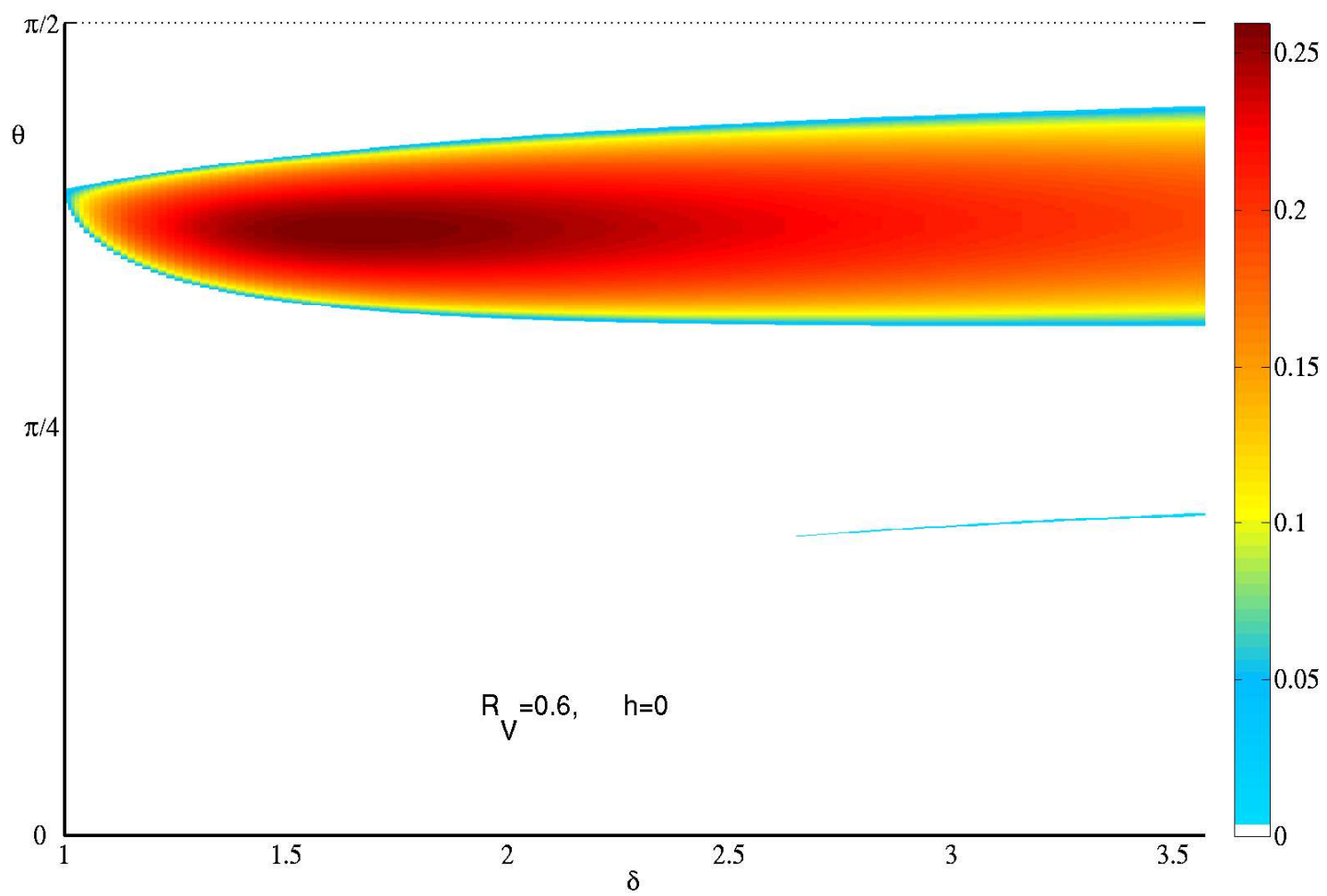
$$\tau = 2\pi$$



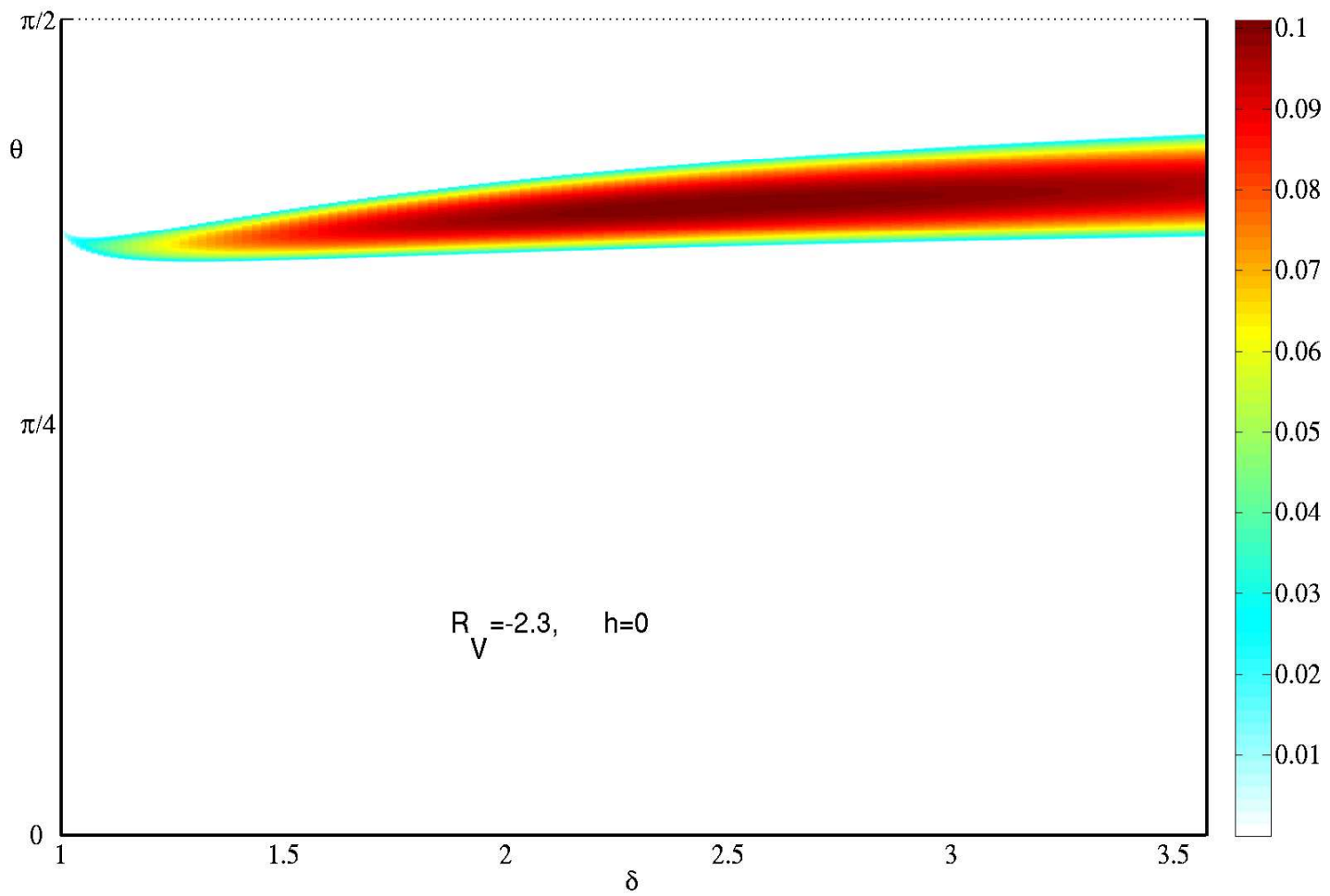
$$(h = 0, \mathcal{R}_v = -0.6).$$



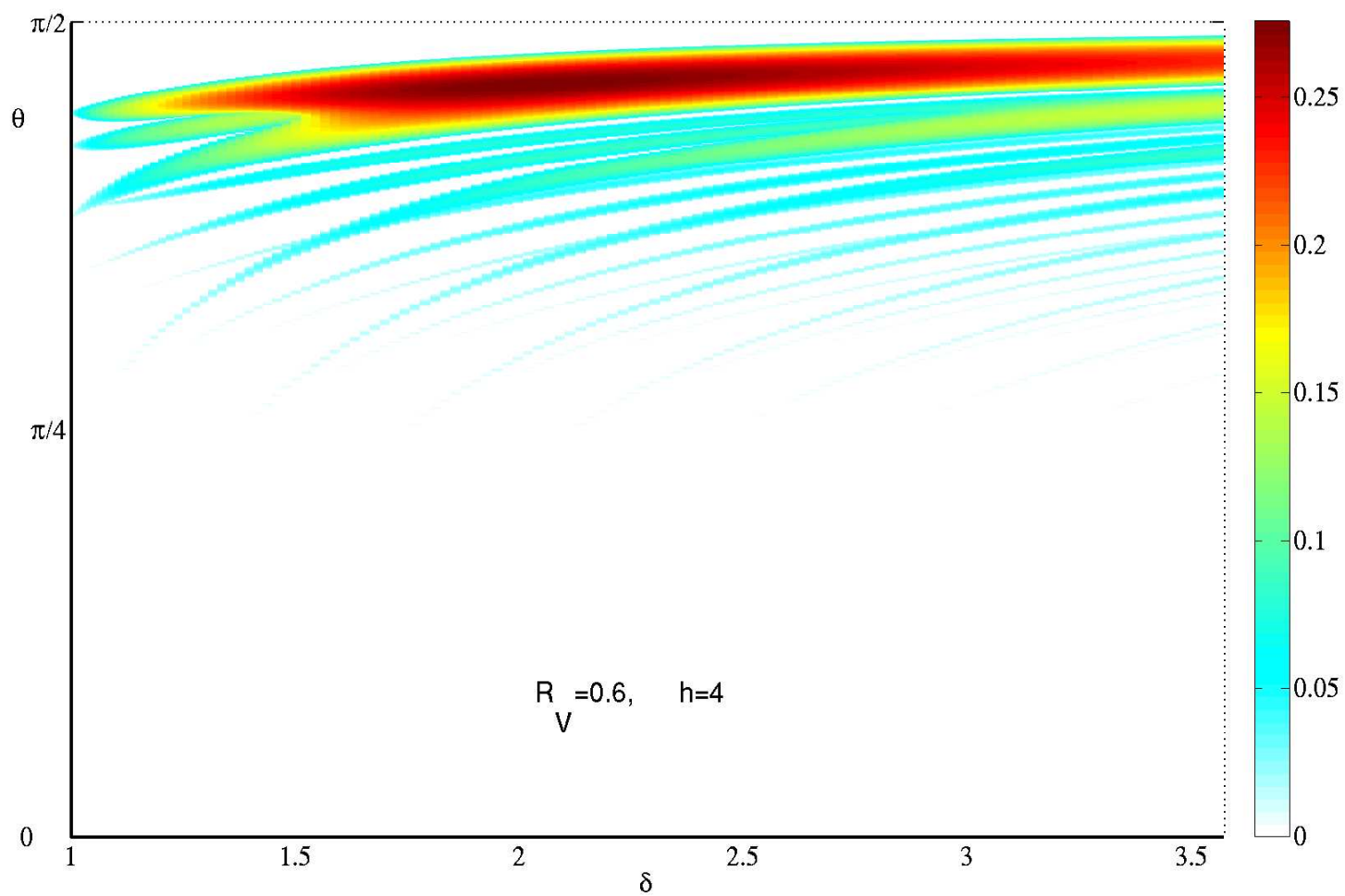
$$(h = 0, \mathcal{R}_v = -1.3).$$



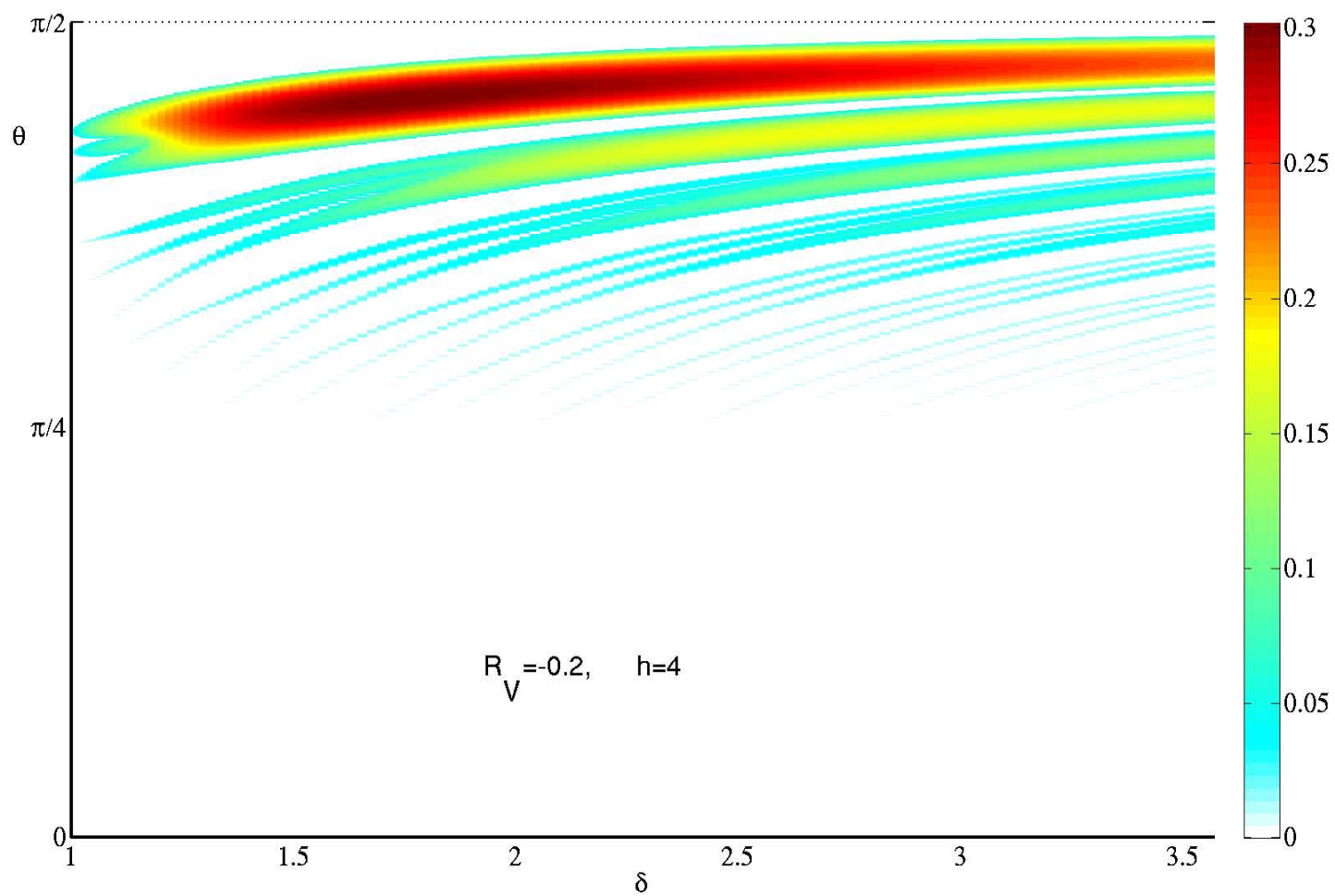
$$(h = 0, \mathcal{R}_v = 0.6).$$



$$(h = 0, \mathcal{R}_v = -2.3).$$



$h = 4$ and $\mathcal{R}_v = 0.6$.



$h = 4$ and $\mathcal{R}_v = -0.2$.