Scaling laws, nonlocality and structure in isotropic magnetohydrodynamic turbulence

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Hydrodynamic turbulence

- Navier-Stokes

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = -\boldsymbol{\nabla} p + \nu \Delta \boldsymbol{u} + \boldsymbol{f}$$
$$\boldsymbol{\nabla} \cdot u = 0$$

- Kolmogorov (1941)

$$\langle \delta u_L^3(r) \rangle - 6\nu \partial_r \langle \delta u_L^2(r) \rangle = -\frac{4}{5}\epsilon r$$ (exact result)

$$\langle \delta u_L^p \rangle \sim \epsilon^{p/3} r^{p/3}$$ (K41 phenomenology)

- Local interactions in *k*-space
- Cascade

Magnetohydrodynamic turbulence

- MHD

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = -\boldsymbol{\nabla} p + \boldsymbol{B} \cdot \boldsymbol{\nabla} \boldsymbol{B} + \nu \Delta \boldsymbol{u} + \boldsymbol{f} \qquad \boldsymbol{\nabla} \cdot u = 0$$

$$\partial_t \boldsymbol{B} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{B} = \boldsymbol{B} \cdot \boldsymbol{\nabla} \boldsymbol{u} + \eta \Delta \boldsymbol{B}$$

Elsässer variables

$$\boldsymbol{z}^{\pm} = \boldsymbol{u} \pm \boldsymbol{B}$$

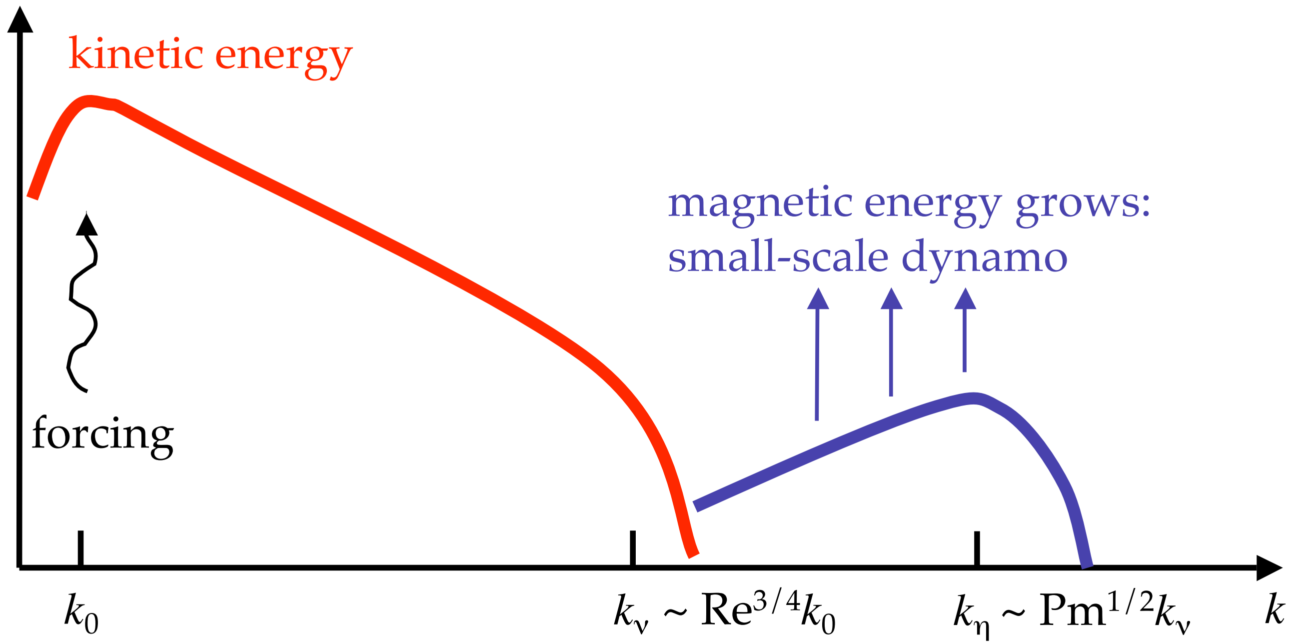
$$\partial_t \boldsymbol{z}^{\pm} + \boldsymbol{z}^{\mp} \cdot \boldsymbol{\nabla} \boldsymbol{z}^{\pm} = -\boldsymbol{\nabla} p + \tfrac{\nu+\eta}{2} \Delta \boldsymbol{z}^{\pm} + \tfrac{\nu-\eta}{2} \Delta \boldsymbol{z}^{\mp} + \boldsymbol{f}$$

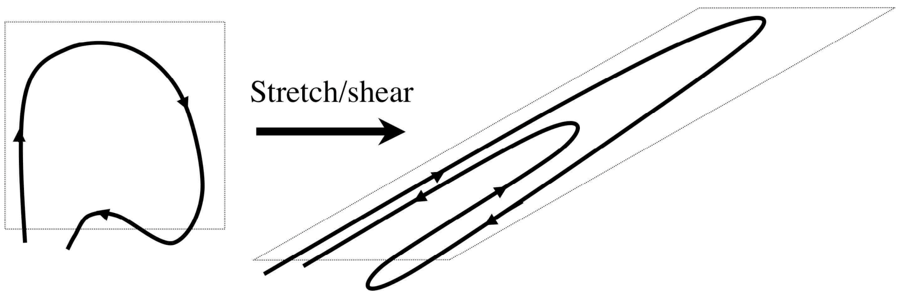
- Exact laws

$$\left\langle \delta u_L^3 \right\rangle - 6 \left\langle B_L^2 \delta u_L \right\rangle - 6\nu \partial_r \left\langle \delta u_L^2 \right\rangle = -\tfrac{4}{5}\, \epsilon r$$

$$\left\langle \delta z_L^{\mp} |\delta \boldsymbol{z}^{\pm}|^2 \right\rangle - \partial_r \left[(\nu + \eta) \left\langle |\delta \boldsymbol{z}^{\pm}|^2 \right\rangle + (\nu - \eta) \left\langle \delta \boldsymbol{z}^{+} \cdot \delta \boldsymbol{z}^{-} \right\rangle \right] = -\tfrac{4}{3}\, \epsilon r$$

Smallscale dynamo





Saturated state:

$$\boldsymbol{B}\cdot\nabla\boldsymbol{B} \sim \boldsymbol{u}\cdot\nabla\boldsymbol{u}$$

resistive scale

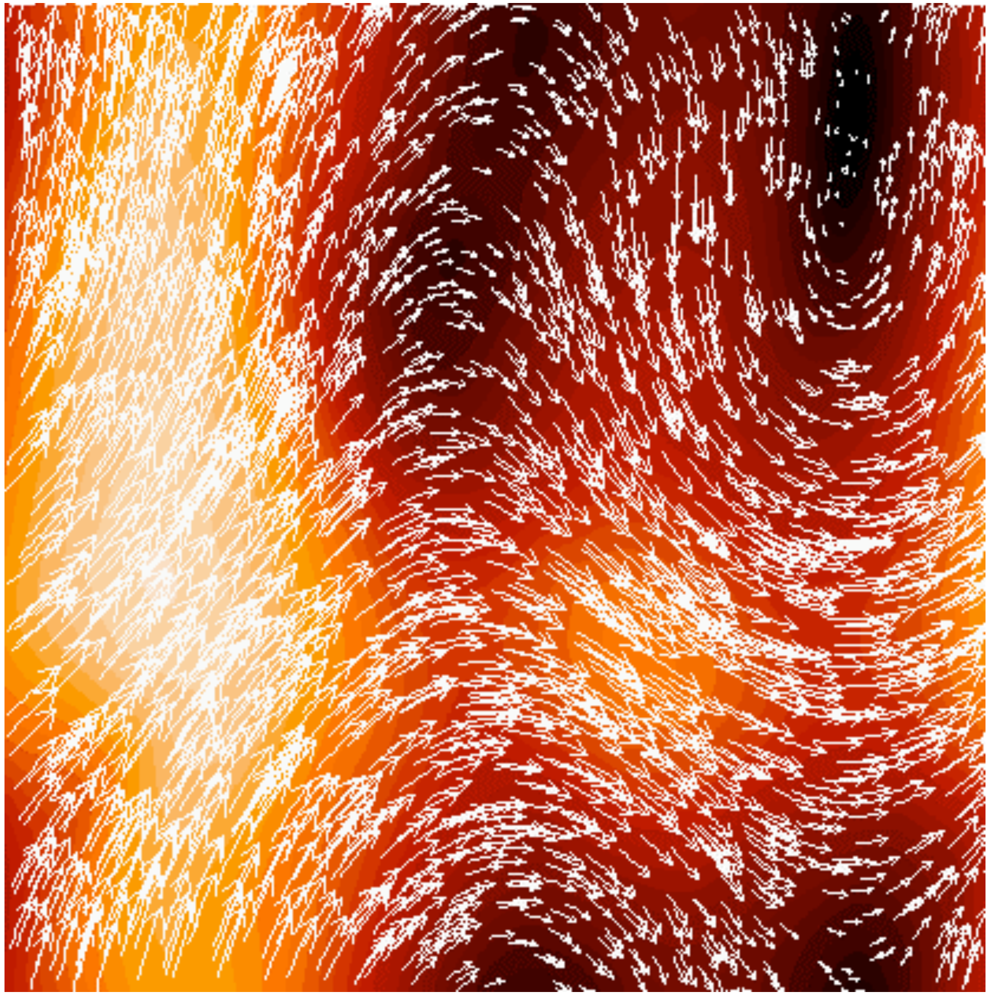
forcing scale

Saturated state is dominated by nonlocal interactions.
How are exact scaling laws satisfied?

Large Pm - Numerical Simulations

- Pr=1250, Re~1 (256^3)

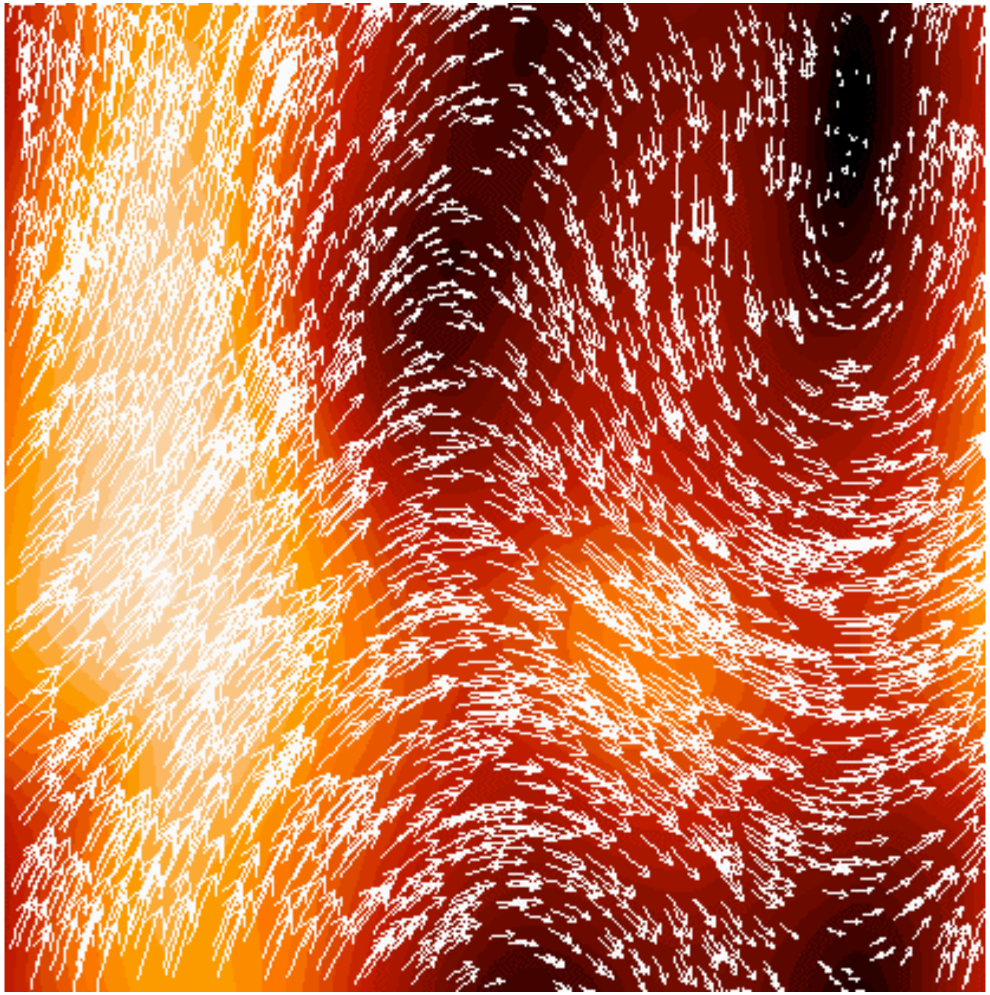
(Schekochihin *et at.* ApJ 2004)



Velocity field: smooth

Magnetic field: stripy

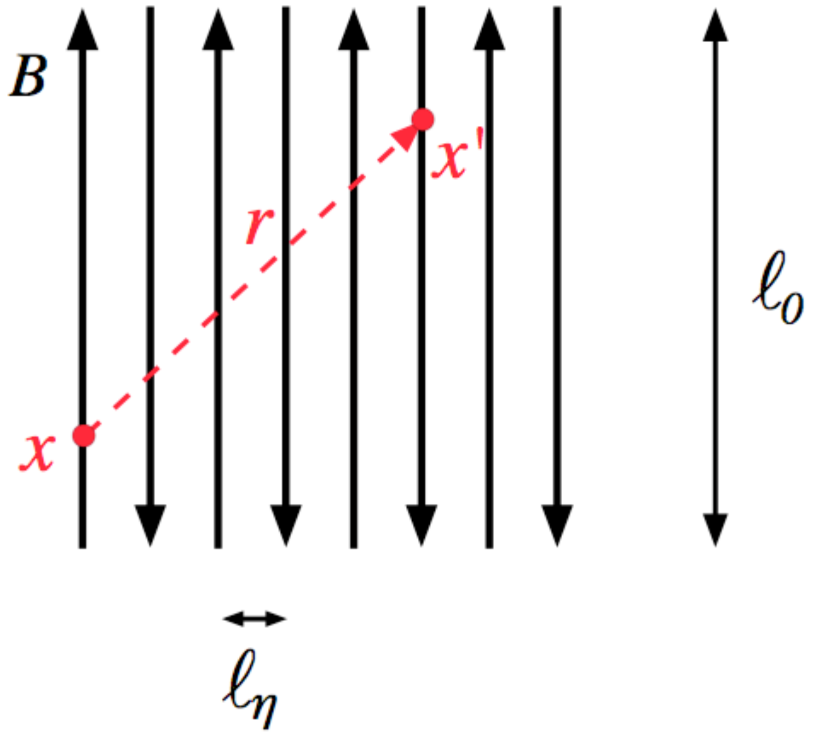
Large Pm - Subviscous regime



Velocity field: smooth

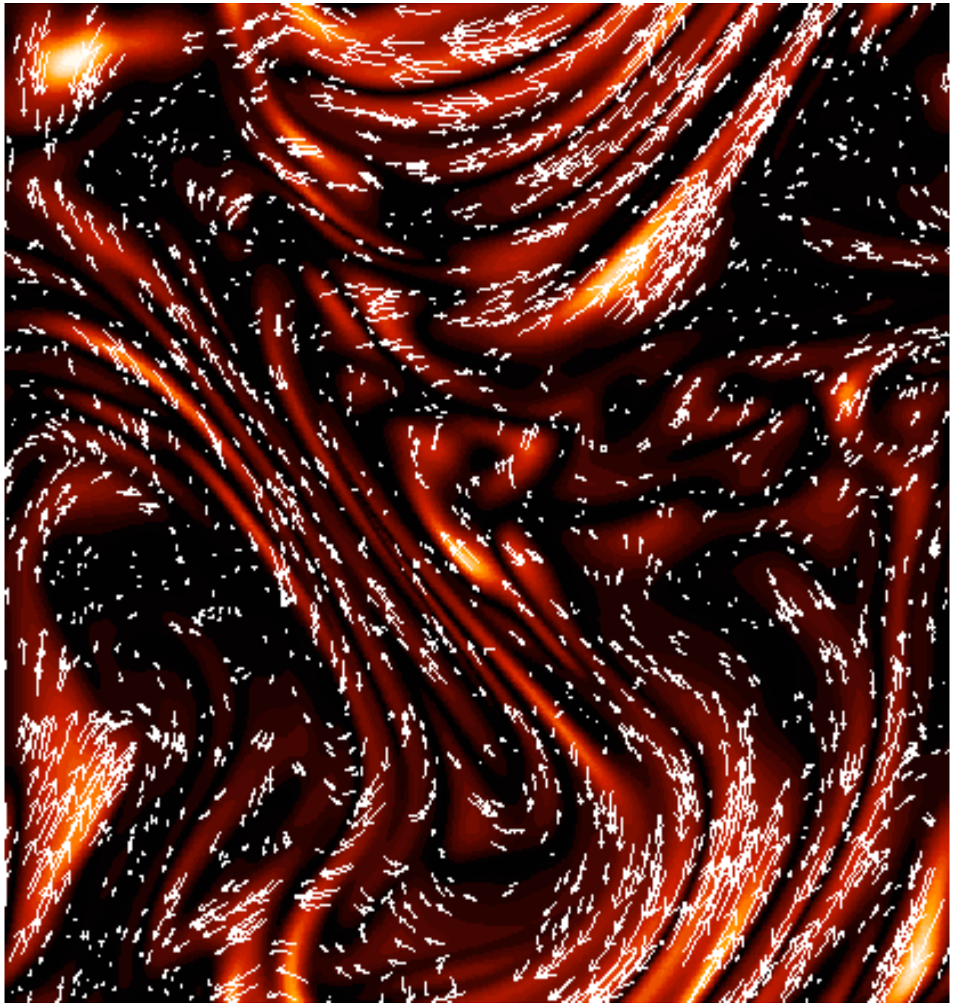
$$\delta \boldsymbol{u} \simeq \boldsymbol{r} \cdot \nabla \boldsymbol{u}(\boldsymbol{x})$$

Large Pm - Stripy field



$\delta \boldsymbol{B} = 0$ probability 1/2

$\delta \boldsymbol{B} = -2\boldsymbol{B}$ probability 1/2



Magnetic field: stripy

Large Pm - Stripy field

4/5 law:

$$\left\langle \delta u_L^3 \right\rangle - 6\left\langle B_L^2 \delta u_L \right\rangle - 6\nu\partial_r \left\langle \delta u_L^2 \right\rangle = -\tfrac{4}{5}\,\epsilon r$$

$$6\left\langle B_L^2 \delta u_L \right\rangle \simeq 6\left\langle B_k B_l \nabla_i u_j \right\rangle \left\langle \hat{r}_i \hat{r}_j \hat{r}_k \hat{r}_l \right\rangle r = \frac{4}{5}\left\langle \boldsymbol{BB} : \boldsymbol{\nabla u} \right\rangle r$$

$$\left\langle \delta u_L^2 \right\rangle \simeq \left\langle (\hat{\boldsymbol{r}}\hat{\boldsymbol{r}} : \boldsymbol{\nabla u})^2 \right\rangle r^2 = \left\langle \nabla_i u_j \nabla_k u_l \right\rangle \left\langle \hat{r}_i \hat{r}_j \hat{r}_k \hat{r}_l \right\rangle r^2 = \frac{1}{15}\left\langle |\boldsymbol{\nabla u}|^2 \right\rangle r^2$$

$$\epsilon = \left\langle \boldsymbol{BB} : \boldsymbol{\nabla u} \right\rangle + \nu\left\langle |\boldsymbol{\nabla u}|^2 \right\rangle = \eta\left\langle |\boldsymbol{\nabla B}|^2 \right\rangle + \nu\left\langle |\boldsymbol{\nabla u}|^2 \right\rangle$$

Large Pm - Stripy field

4/3 law:

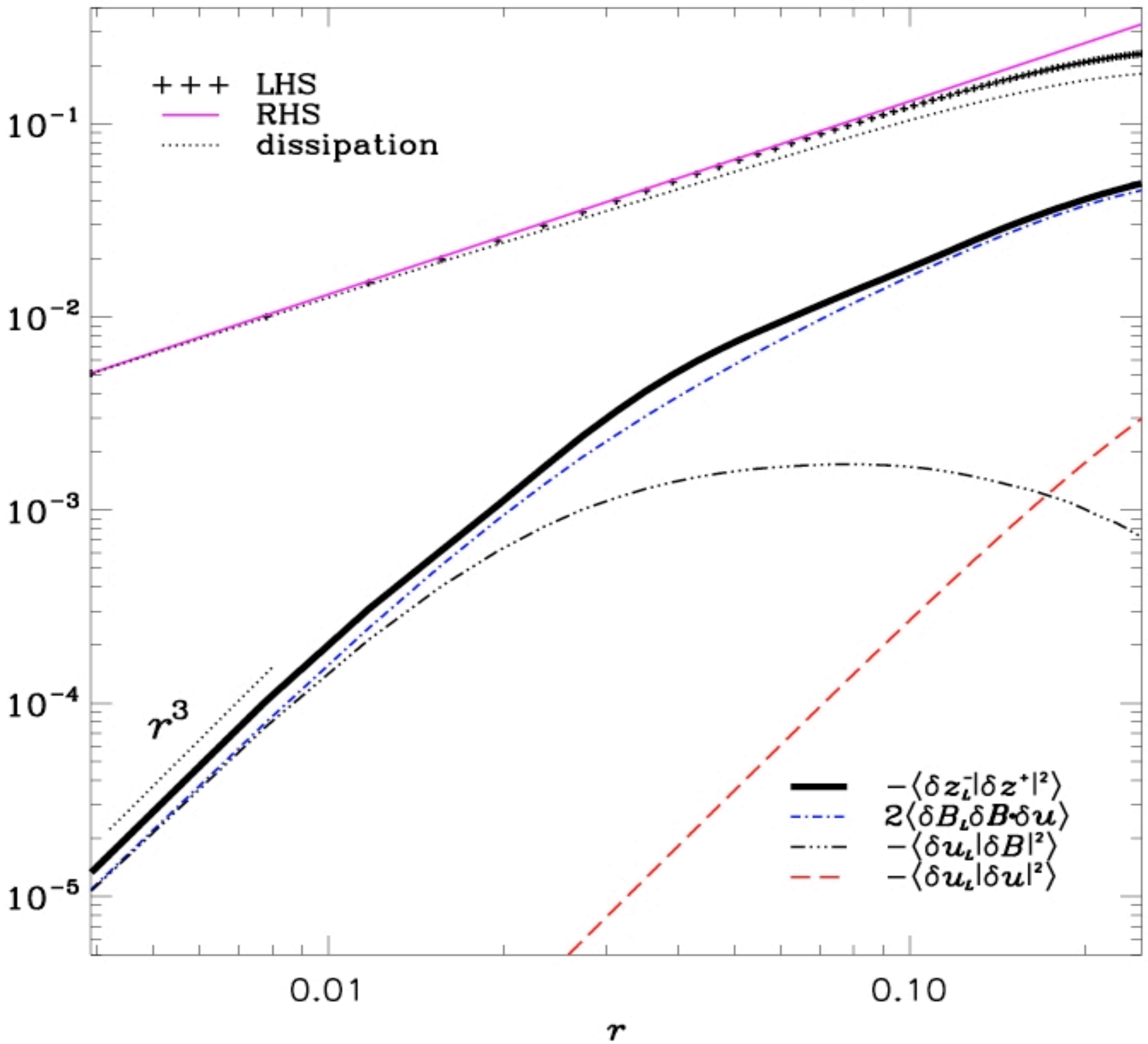
$$\begin{aligned}\left\langle \delta z_L^{\mp} |\delta \boldsymbol{z}^{\pm}|^2 \right\rangle &= \left\langle \delta u_L |\delta \boldsymbol{u}|^2 \right\rangle \pm 2\left\langle \delta u_L \delta \boldsymbol{u}\cdot\delta \boldsymbol{B} \right\rangle \mp \left\langle \delta B_L |\delta \boldsymbol{u}^2| \right\rangle \\ &+ \left\langle \delta u_L |\delta \boldsymbol{B}|^2 \right\rangle - 2\left\langle \delta B_L \delta \boldsymbol{u}\cdot\delta \boldsymbol{B} \right\rangle \mp \left\langle \delta B_L |\delta \boldsymbol{B}|^2 \right\rangle \\ &= -\frac{4}{3}\epsilon r + 2\partial_r \left[\nu \left\langle |\delta \boldsymbol{u}|^2 \right\rangle \pm (\nu+\eta)\left\langle \delta \boldsymbol{u}\cdot\delta \boldsymbol{B} \right\rangle + \eta \left\langle |\delta \boldsymbol{B}|^2 \right\rangle \right].\end{aligned}$$

$$\begin{aligned}\left\langle \delta u_L |\delta \boldsymbol{B}|^2 \right\rangle &= 2\left\langle B^2 \nabla_i u_j \right\rangle \left\langle \hat{r}_i \hat{r}_j \right\rangle r = \frac{2}{3}\left\langle B^2 \boldsymbol{\nabla}\cdot\boldsymbol{u} \right\rangle r = 0, \\ 2\left\langle \delta B_L \delta \boldsymbol{u}\cdot\delta \boldsymbol{B} \right\rangle &= 4\left\langle B_i B_j \nabla_k u_j \right\rangle \left\langle \hat{r}_i \hat{r}_k \right\rangle r = \frac{4}{3}\left\langle \boldsymbol{B}\boldsymbol{B} : \boldsymbol{\nabla}\boldsymbol{u} \right\rangle r, \\ \left\langle \delta B_L |\delta \boldsymbol{B}|^2 \right\rangle &= -4\left\langle B^2 B_i \right\rangle \left\langle \hat{r}_i \right\rangle = 0,\end{aligned}$$

$$\begin{aligned}\left\langle |\delta \boldsymbol{u}|^2 \right\rangle &= \left\langle \nabla_i u_j \nabla_k u_j \right\rangle \left\langle \hat{r}_i \hat{r}_k \right\rangle r^2 = \frac{1}{3}\left\langle |\boldsymbol{\nabla}\boldsymbol{u}|^2 \right\rangle r^2, \\ \left\langle \delta \boldsymbol{u}\cdot\delta \boldsymbol{B} \right\rangle &= -\left\langle B_i \nabla_j u_i \right\rangle \left\langle \hat{r}_j \right\rangle r = 0, \\ \left\langle |\delta \boldsymbol{B}|^2 \right\rangle &= 2\left\langle B^2 \right\rangle.\end{aligned}$$

$$\epsilon = \left\langle \boldsymbol{B}\boldsymbol{B} : \boldsymbol{\nabla}\boldsymbol{u} \right\rangle + \nu \left\langle |\boldsymbol{\nabla}\boldsymbol{u}|^2 \right\rangle = \eta \left\langle |\boldsymbol{\nabla}\boldsymbol{B}|^2 \right\rangle + \nu \left\langle |\boldsymbol{\nabla}\boldsymbol{u}|^2 \right\rangle$$

Large Pm

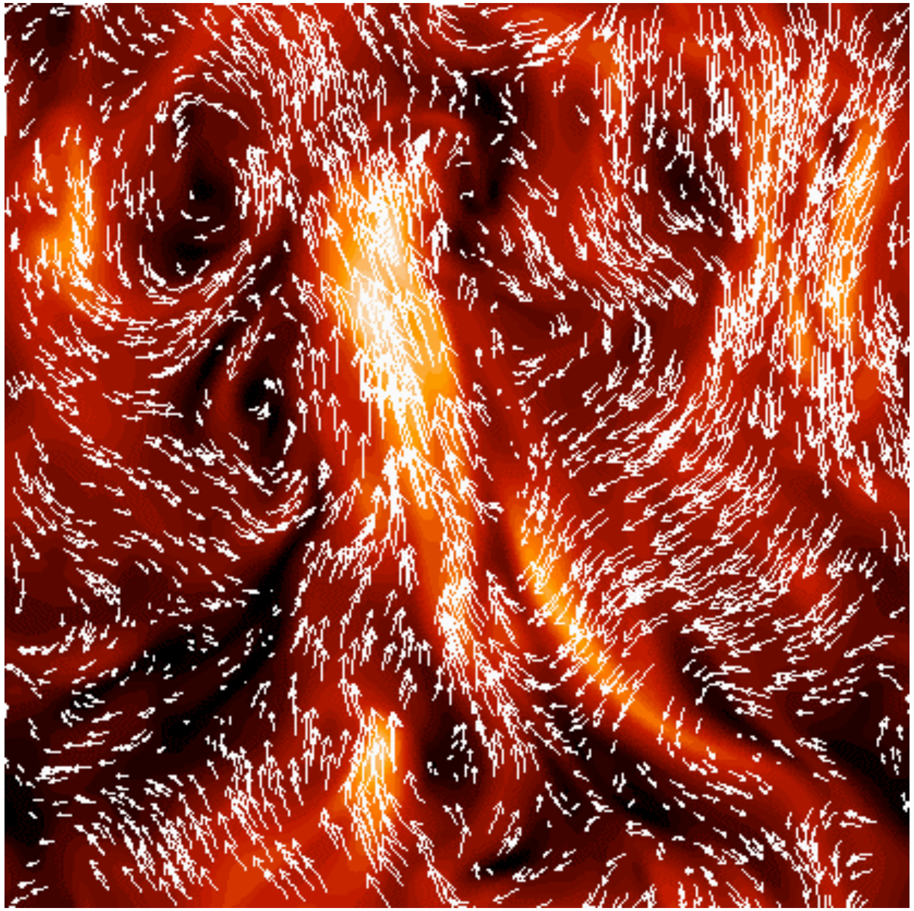


$$\left\langle \delta z_L^{\mp} |\delta \boldsymbol{z}^{\pm}|^2 \right\rangle = \left\langle \delta u_L |\delta \boldsymbol{u}|^2 \right\rangle \pm 2 \left\langle \delta u_L \delta \boldsymbol{u} \cdot \delta \boldsymbol{B} \right\rangle \mp \left\langle \delta B_L |\delta \boldsymbol{u}^2| \right\rangle + \left\langle \delta u_L |\delta \boldsymbol{B}|^2 \right\rangle - 2 \left\langle \delta B_L \delta \boldsymbol{u} \cdot \delta \boldsymbol{B} \right\rangle \mp \left\langle \delta B_L |\delta \boldsymbol{B}|^2 \right\rangle$$

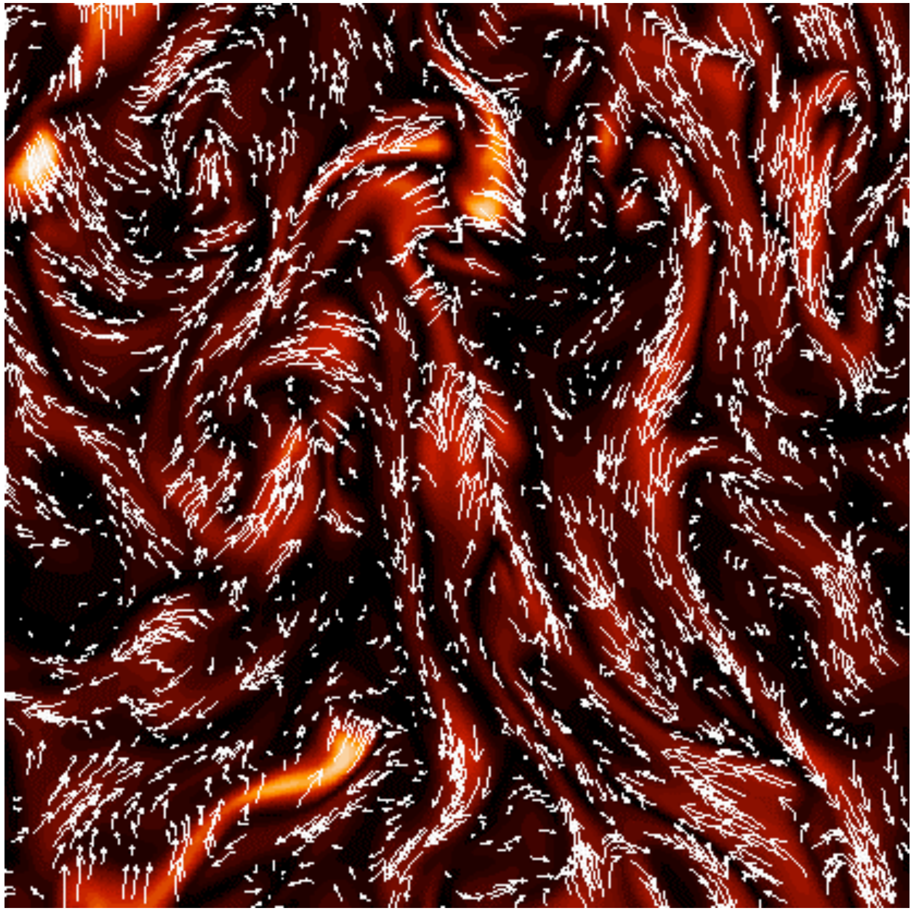
$$\left\langle \delta z_L^{\mp} |\delta \boldsymbol{z}^{\pm}|^2 \right\rangle - \partial_r \left[(\nu + \eta) \left\langle |\delta \boldsymbol{z}^{\pm}|^2 \right\rangle + (\nu - \eta) \left\langle \delta \boldsymbol{z}^{+} \cdot \delta \boldsymbol{z}^{-} \right\rangle \right] = -\frac{4}{3} \epsilon r$$

Pm~1 - Numerical Simulations

- Pm=1, Re=400. (256^3)



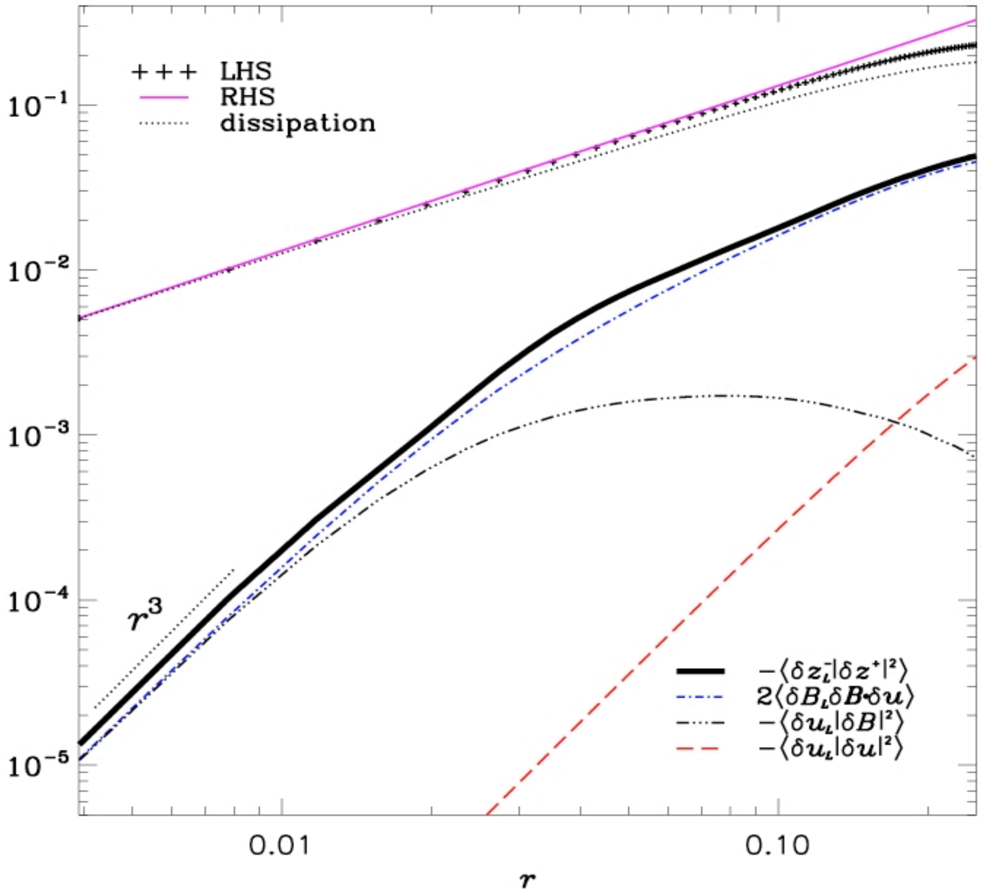
Velocity field



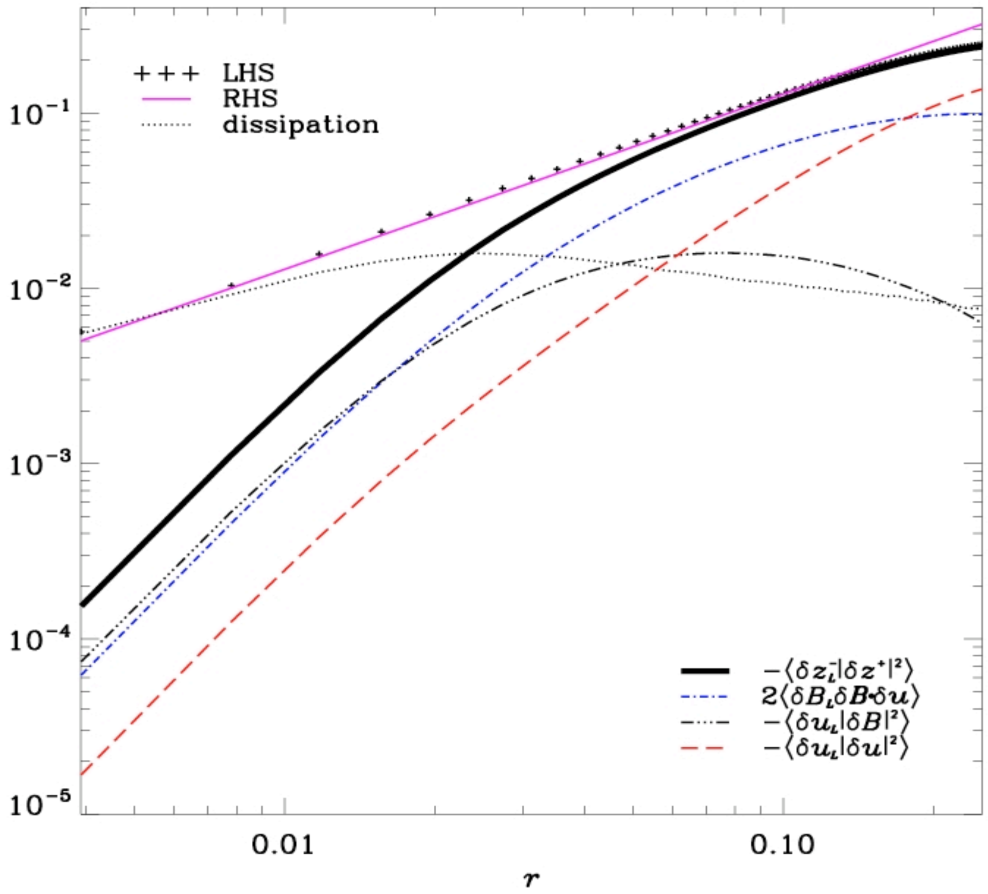
Magnetic field

Pm~1

Pm~1250



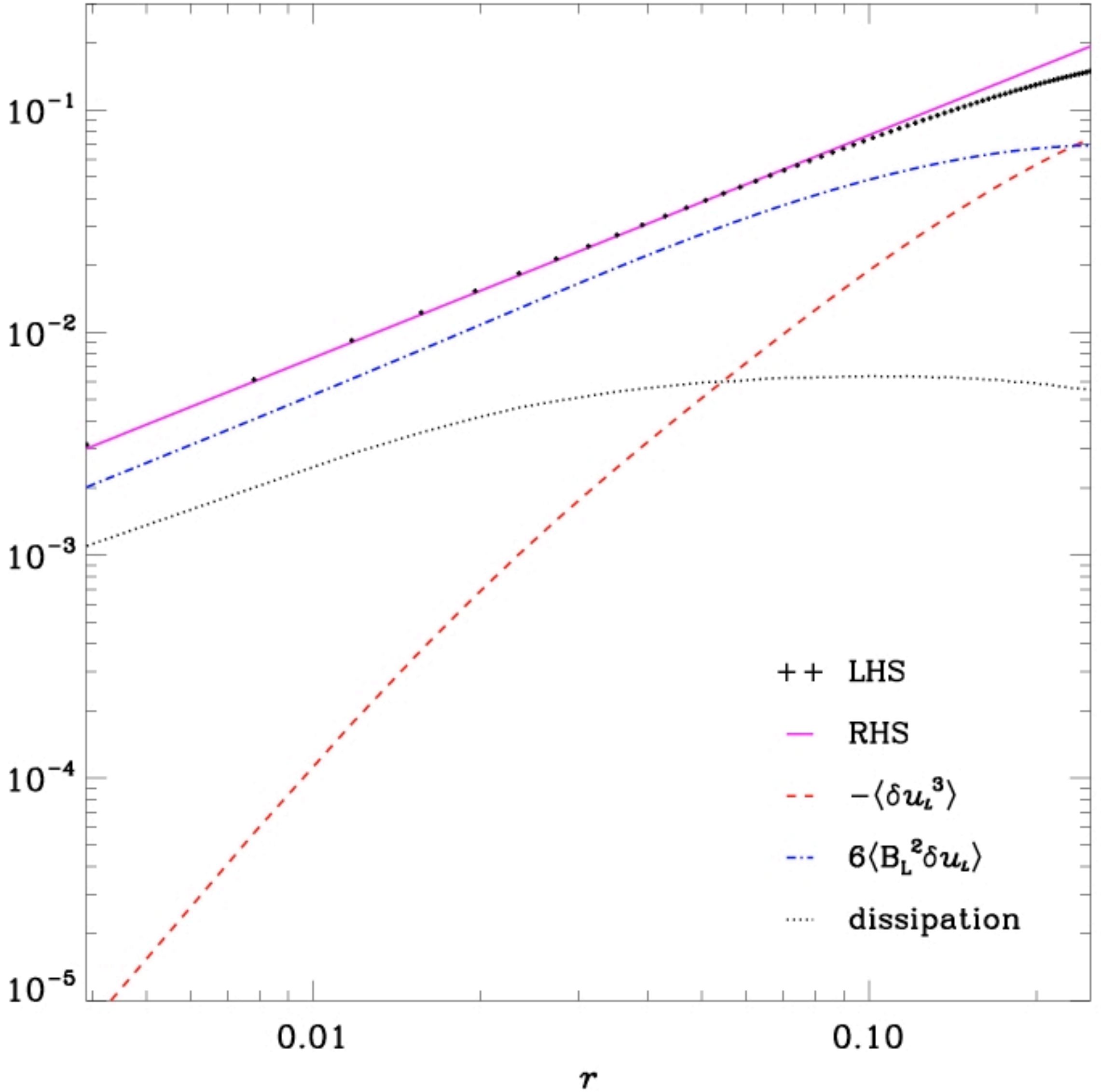
Pm~1



$$\left\langle \delta z_L^{\mp} |\delta \boldsymbol{z}^{\pm}|^2\right\rangle - \partial_r \left[(\nu+\eta)\left\langle|\delta\boldsymbol{z}^\pm|^2\right\rangle + (\nu-\eta)\left\langle\delta\boldsymbol{z}^+\cdot\delta\boldsymbol{z}^-\right\rangle\right] = -\frac{4}{3}\epsilon r$$

$$\left\langle \delta z_L^{\mp}|\delta \boldsymbol{z}^\pm|^2\right\rangle = \left\langle\delta u_L|\delta\boldsymbol{u}|^2\right\rangle \pm 2\left\langle \delta u_L \delta\boldsymbol{u}\cdot\delta\boldsymbol{B}\right\rangle \mp \left\langle\delta B_L|\delta \boldsymbol{u}^2|\right\rangle + \left\langle \delta u_L|\delta\boldsymbol{B}|^2\right\rangle - 2\left\langle\delta B_L\delta\boldsymbol{u}\cdot\delta\boldsymbol{B}\right\rangle \mp\left\langle\delta B_L|\delta\boldsymbol{B}|^2\right\rangle$$

Pm~1



$$\left\langle \delta u_L^3 \right\rangle - 6 \left\langle B_L^2 \delta u_L \right\rangle - 6\nu\partial_r \left\langle \delta u_L^2 \right\rangle = -\frac{4}{5}\,\epsilon r$$

Conclusions

- Even though certain triple correlation functions scale linearly with *r* stationary isotropic MHD turbulence it is controlled by nonlocal interactions between forcing-scale motions and resistive-scale folded magnetic structures.
- There is no energy cascade. The kinetic energy cascade is short-circuited by energy transfer from forcing scale motions directly into the small scale magnetic field.
- The linear scalings prescribed by the exact scaling laws are satisfied by organizing the magnetic field into a folded structure with direction reversals at the resistive scale.
- This emphasizes that the existence of exact scaling laws is by itself not a proof of a local cascade.