Transition to turbulence in channel flow with spanwise magnetic field

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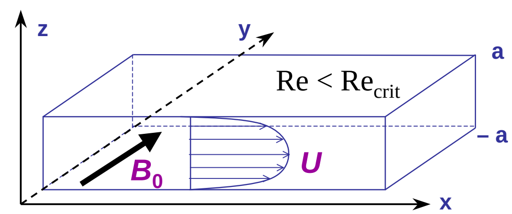
MHD flow with spanwise magnetic field

Channel geometry



$$Re = \frac{aU}{\nu},$$

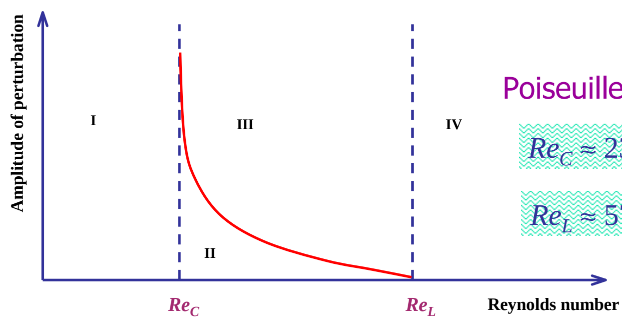
$$Ha = B_0 a \sqrt{\frac{\sigma}{\rho\nu}}$$

$$N = \frac{Ha^2}{Re} = \frac{B_0^2 a\sigma}{\rho U}$$

Question:

How does the transition to turbulence occur in the presence of a spanwise magnetic field?

Stability of shear flows



Poiseuille flow:

$Re_C \approx 2300$

$Re_L \approx 5772$

- Globally stable (initial growth of perturbations is possible)
- Conditionally stable: initial growth, then asymptotic decay
- Conditionally unstable: instability will occur
- Linearly unstable

Transition scenario (non-MHD case)

- Basic flow + 2D streamwise vortices (optimal disturbance: strong growth for finite time)
- At time t_{opt} of maximum growth: impose small 3D random perturbation

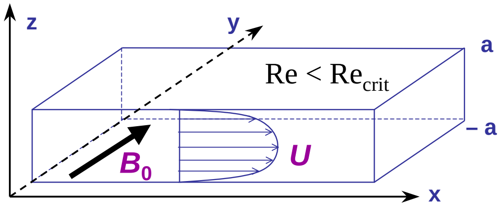
$$\boldsymbol{V} = U(z)\boldsymbol{e}_x + \boldsymbol{V}_2(y,z,t) + \varepsilon \boldsymbol{V}_3(x,y,z,t)$$

- Route to transition:

2D streamwise vortices ⇒ Evolution into streaks ⇒ 3D random perturbations ⇒ Streak breakdown leads to transition

(*streak breakdown*, Henningson, Reddy, Schmid)

Governing equations



$$Re = \frac{aU}{\nu},$$

$$Ha = B_0 a \sqrt{\frac{\sigma}{\rho \nu}}$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p + \frac{1}{Re}\Delta \mathbf{v} + \frac{Ha^2}{Re}(\mathbf{j} \times \mathbf{e}_0)$$

$$\mathbf{j} = -\nabla \Phi + \mathbf{v} \times \mathbf{e}_0, \ \nabla \cdot \mathbf{j} = 0, \ \Delta \Phi = \nabla \cdot (\mathbf{v} \times \mathbf{e}_0)$$

In linearized problem solutions are Fourier modes with respect to homogeneous directions x and y -> streamwise and spanwise wavenumbers α and β

Linear problem formulation

Linear evolution equations formulated for the following variables:
η – vertical vorticity, w – vertical velocity, Φ – electric potential

$$1.\ \partial_t \eta + i\alpha U \eta - (D^2 - k^2)\eta / \mathrm{Re} = i\beta w DU - i\beta N\left[D\Phi - (i\alpha Dw + i\beta\eta)/k^2\right]$$

$$2.\ \left[\partial_t + i\alpha U - (D^2 - k^2)/\mathrm{Re}\right](D^2 - k^2)w = i\alpha w D^2 U + \beta^2 N w$$

$$3.\ (D^2 - k^2)\Phi = (i\alpha D^2 w + i\beta D\eta)/k^2 - i\alpha w$$

$$k^2 = \alpha^2 + \beta^2,\ D \equiv \partial/\partial z,\ N = Ha^2/\mathrm{Re}$$

perturbation energy

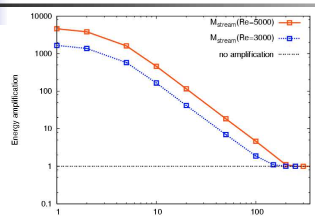
$$E(t) = \int_{-1}^{1} \left(|Dw|^2 + k^2 |w|^2\right) dz + \beta^2 \mathrm{Re}^2 \int_{-1}^{1} |\eta|^2 dz$$

Energy amplification

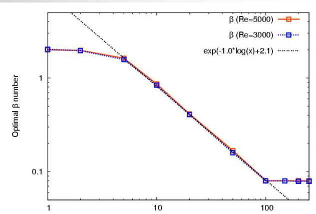
$$G(t, \alpha, \beta, Ha, \mathrm{Re}) = \max |E(t)| / |E(0)|,$$

$$M(t, Ha, \mathrm{Re}) = \sup_{\alpha,\beta} |G(t, \alpha, \beta, Ha, \mathrm{Re})|$$

Optimal streamwise vortices: damping of transient growth by magnetic field

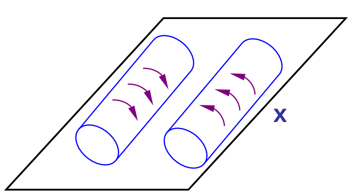


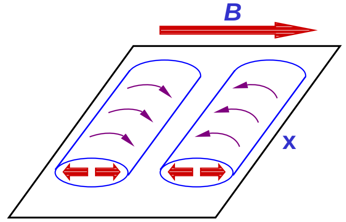
Energy amplification M vs. Ha for Reynolds numbers Re = 3000 and 5000



Optimal spanwise wavenumber β vs. Ha for Reynolds numbers Re = 3000 and 5000

No field





Arguments for the scaling of streamwise vortices

Linear results yield: $M_{stream} \approx Ha^{-2}$ and $\beta_{stream} \approx Ha^{-1}$ for $5 < Ha < 100$

New scaled variables: $t' = t/\mathrm{Re},\ \eta' = \eta/(\beta \mathrm{Re}),\ w' = w,\ \Phi' = \Phi/\mathrm{Re}$

We find that $\partial_t D\eta' - (D^2 - \beta^2)D\eta' = iD(w'DU) - i\beta^2 Ha^2 \Phi'$

Squire equation is responsible for the energy amplification – as in ordinary hydrodynamics
$\Rightarrow M_{stream}(\beta, Ha, Re) = \beta^2_{stream} Re^2 F(R)$, $R = \beta^2_{stream} Ha^2$ and $F(R)$ is the amplification given by:

$$\partial_t D\eta' - D^2 D\eta' = iD(w'DU) - iR\Phi'$$

$$\left[\partial_t - D^2\right]D^2 w' = Rw$$

$$D^2\Phi' = iD\eta'$$

Let $F(R)$ have a maximum at $R = R_c$ and decrease as $1/R^p$ with $p > 1$ for $R \to \infty$

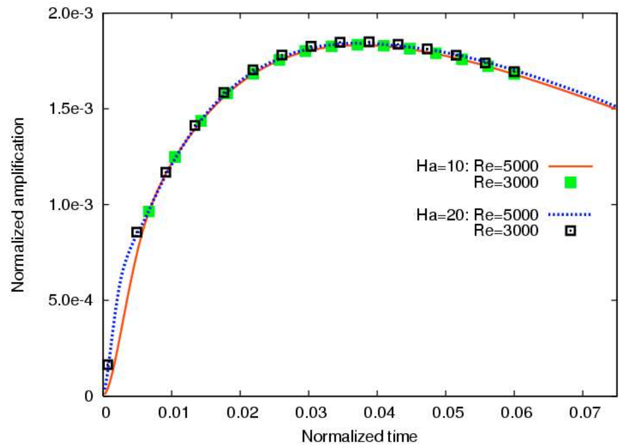
$\Rightarrow$ Three possible cases

1. If $\beta^2_{stream} \approx 1/Ha^n$ with $n > 1$ then: $R \to 0$ and $M_{stream} \to Re^2/Ha^{2n} F(0)$

2. If $\beta^2_{stream} \approx 1/Ha^n$ with $0 < n < 1$ then: $R \to \infty$ and $M_{stream} = Re^2 \beta^2_{stream} F(R \to \infty) << Re^2/Ha^2$
$\Rightarrow$ decreases faster than $1/Ha^2$

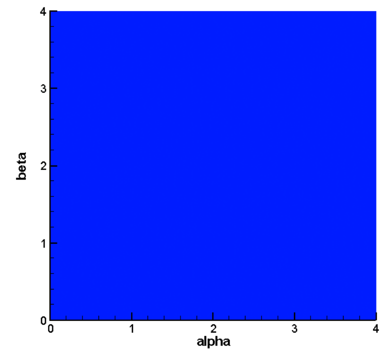
3. If $\beta^2_{stream} \approx 1/Ha$ then: $R = const$ and $M_{stream} = R_c F(R_c) Re^2/Ha^2$
$\Rightarrow$ hypothesis **3** is correct since it results in the observed scaling

Streamwise vortices: damping and scaling

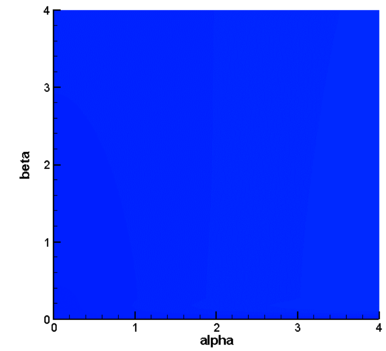


Energy amplification *M* vs. time *T* for *Re* = 3000 & 5000 and *Ha* = 10 & 20: in rescaled units nearly collapse into one curve

General case: perturbations of arbitrary orientation



***Ha* = 10 & *Re* = 5000**

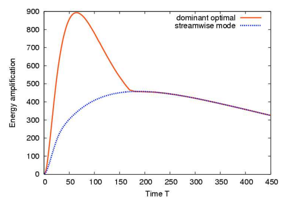


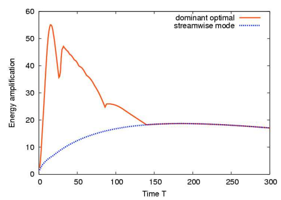
***Ha* = 50 & *Re* = 5000**

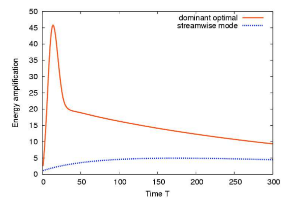
Isolevels of energy amplification *M* (α, β, *T*) in the entire (α, β) domain

Change of energy amplification and optimal wavenumbers vs. time *T* and *Ha* at *Re* = 5000

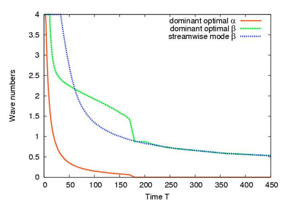
Energy amplification *M* for global and streamwise optimal modes



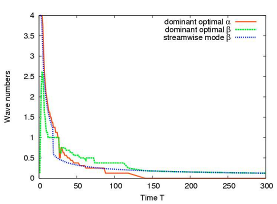




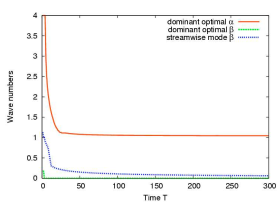
Optimal wavenumbers (α, β) for global and streamwise modes



***Ha* = 10**

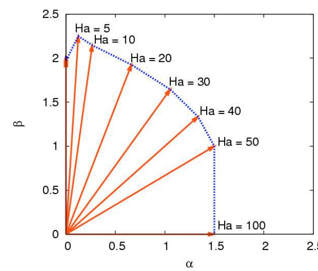


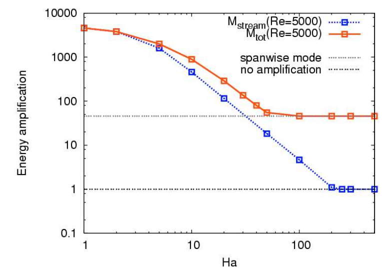
***Ha* = 50**

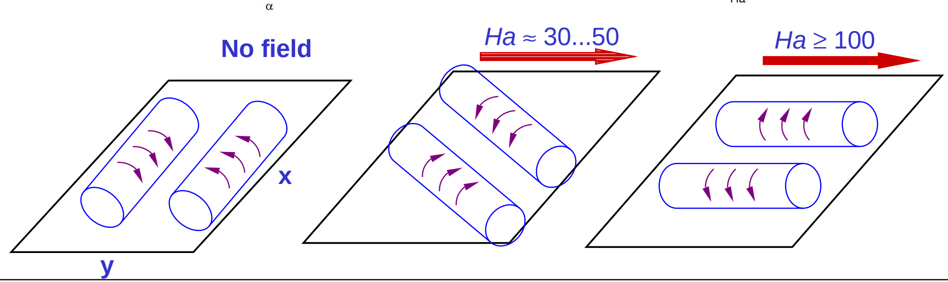


***Ha* = 100**

Optimal oblique modes at *Re* =5000: summary

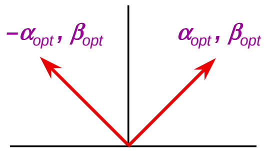




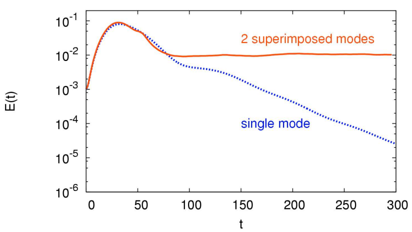


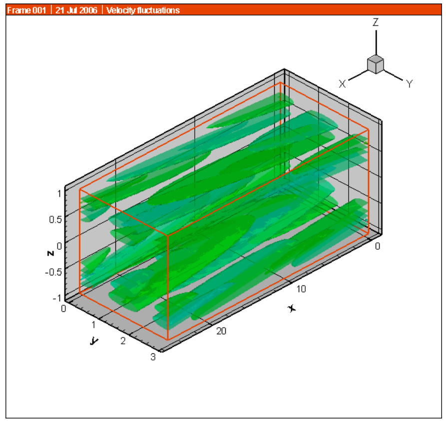
Superposition of optimal oblique modes as transition scenario

Optimal (α, β) mode and its symmetric ($-\alpha$, β) counterpart provide the same linear growth



Non-linear interaction of superimposed oblique modes as a source of instability

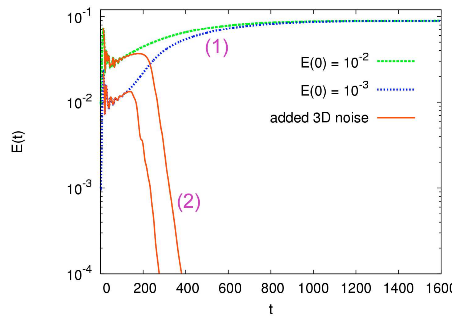




Transition to turbulence by non-linear interaction of two superimposed oblique modes. *Re* = 5000 and *Ha* = 10, iso-surfaces of streamwise velocity perturbations

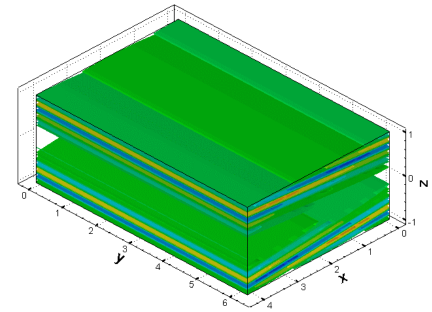
Transition at high Hartmann numbers ($Ha \geq 100$)

At high *Ha* purely 2D Orr-modes ($\alpha \neq 0$, $\beta = 0$) become optimal: no interaction with spanwise magnetic field



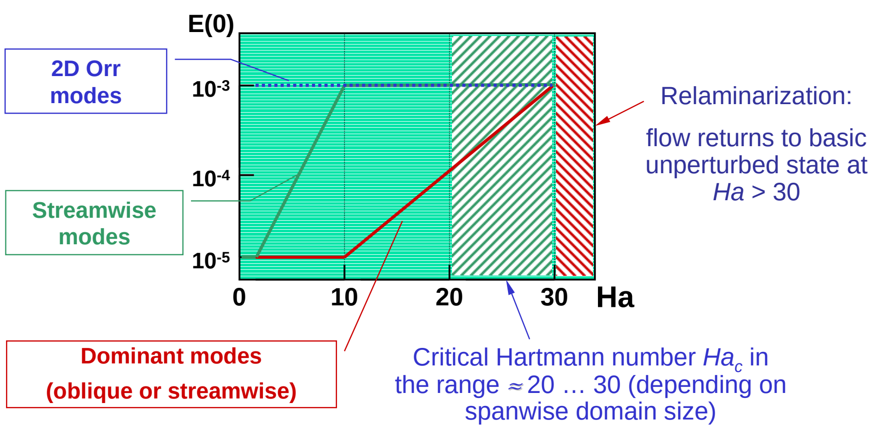
Evolution of Orr-modes at *Re* = 5000 and *Ha* = 100

(1) If no noise added: flow remains purely 2D
(2) 3D noise: triggers re-laminarization – Joule dissipation drains energy from 2D modes



Flow re-laminarization and return to unperturbed state triggered by 3D noise. *Re* = 5000 and *Ha* = 100, iso-surfaces of streamwise velocity perturbations

Stability thresholds for initial energy of optimal perturbations *E(0)* versus *Ha*



Summary and outlook

Summary:

- streamwise modes strongly damped by magnetic field
- highest energy amplification for larger Ha for oblique modes
- oblique modes align with B as Ha increases
- superposition of symmetric oblique modes favorable for transition
- relaminarization of 2d time-dependent flow at high Ha

Things to do:

- classification of oblique modes
- transition studies with high aspect ratios
- comparison with experiments?

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