# Multiscale Analysis of Convective Magnetic Systems in a Horizontal Layer

Manuel Alexandre Vieira Baptista

CMUP
Centro de Matemática
Universidade do Porto

Max Planck Institute for Polymer Research
Mainz, Germany

ETC11 - 11th European Turbulence Conference
25th June 2007

# Outline

- Motivation
- Convection in MHD systems
- Multiscale Analysis
- Numerical Methods and Programming
- Numerical Results
- Summary / Further Work

# Motivation

- Magnetic fields associated to astrophysical and geophysical bodies are due to electrically conducting fluids (plasmas)
- The magnetohydrodynamic (MHD) approximation describes the dynamics of plasmas with small characteristic flow velocity
- A simplified version of Maxwell equations and the Navier-Stokes can be used to simulate these plasmas

# Motivation

- Magnetic fields associated to astrophysical and geophysical bodies are due to electrically conducting fluids (plasmas)
- The magnetohydrodynamic (MHD) approximation describes the dynamics of plasmas with small characteristic flow velocity
- A simplified version of Maxwell equations and the Navier-Stokes can be used to simulate these plasmas

- Accurate simulations for geo- and astrophysical parameter values are close to impossible
  - limited cpu power $\longrightarrow$ inadequate temporal resolution
  - limited physical memory $\longrightarrow$ inadequate spatial resolution
- Since no analytic solutions are known, a semi-analytic approach seems reasonable

# Motivation

- Eddy viscosity is used to model the dynamics of eddies in turbulent flows, in the same way that molecular viscosity is used to model molecular dynamics.

$$\partial_t \mathbf{V} = \mathbf{V} \times (\partial \times \mathbf{V}) - \partial p + \nu \partial^2 \mathbf{V}$$

- In a naive approximation eddy viscosity is a scalar parameter containing the short-scale details, whose value can be obtained empirically.

$$\partial_t \mathbf{V}_L = \mathbf{V}_L \times (\partial \times \mathbf{V}_L) - \partial p_L + \nu_e \partial^2 \mathbf{V}_L$$

# Motivation

- Eddy viscosity is used to model the dynamics of eddies in turbulent flows, in the same way that molecular viscosity is used to model molecular dynamics.

$$\partial_t \mathbf{V} = \mathbf{V} \times (\partial \times \mathbf{V}) - \partial p + \nu \partial^2 \mathbf{V}$$

- In a naive approximation eddy viscosity is a scalar parameter containing the short-scale details, whose value can be obtained empirically.

$$\partial_t \mathbf{V}_L = \mathbf{V}_L \times (\partial \times \mathbf{V}_L) - \partial p_L + \nu_e \partial^2 \mathbf{V}_L$$

- Multiscale analysis can be used to model eddy viscosity:
  - large scales can be solved analytically, but the solutions depend on short-scales
  - the general solution of short-scales requires numerical methods

# Motivation

- In hydrodynamic turbulent systems eddy viscosity can be negative (and even complex)
  - B. Dubrulle and U. Frisch, Eddy vsicosity of parity-invariant flow, Phys. Rev. A, 43: 5355, 1991.
  - M. Vergassola, S. Gama, and U. Frisch, Proving the existence of negative isotropic eddy-viscosity, In Proceed. NATO-ASI: Theory of Solar and Planetary Dynamos, pages 321–327, Cambridge University Press, 1993.
  - A. Wirth, Complex eddy viscosity: a three-dimensional effect, Physica D, 76: 312, 1994.
  - L. Biferale, A. Crisanti, M. Vergassola and A. Vulpiani, Eddy diffusivities in scalars transport, Phys. Fluids 7: 2725, 1995.

# Motivation

- In hydrodynamic turbulent systems eddy viscosity can be negative (and even complex)
  - B. Dubrulle and U. Frisch, Eddy vsicosity of parity-invariant flow, Phys. Rev. A, 43: 5355, 1991.
  - M. Vergassola, S. Gama, and U. Frisch, Proving the existence of negative isotropic eddy-viscosity, In Proceed. NATO-ASI: Theory of Solar and Planetary Dynamos, pages 321–327, Cambridge University Press, 1993.
  - A. Wirth, Complex eddy viscosity: a three-dimensional effect, Physica D, 76: 312, 1994.
  - L. Biferale, A. Crisanti, M. Vergassola and A. Vulpiani, Eddy diffusivities in scalars transport, Phys. Fluids 7: 2725, 1995.

- In dynamo theory, negative eddy diffusivity is a possible dynamo mechanism
  - A. Lanotte, A. Noullez, M. Vergassola, and A. Wirth, Large-scale dynamo produced by negative magnetic eddy diffusivities, Geophys. Astrophys. Fluid Dynam., 91: 131, 1999.
  - V. A. Zheligovsky, O. M. Podvigina, and U. Frisch, Dynamo effect in parity-invariant flow with large and moderate separation of scales, Geophys. Astrophys. Fluid Dynam., 95: 227, 2001.

# Motivation

**Is it possible to extend these notions to Convective Hydromagnetic (CHM) systems?**

# Motivation

**Is it possible to extend these notions to Convective Hydromagnetic (CHM) systems?**

- The velocity field obeys the Navier-Stokes equation with Lorentz force
- Convection is taken into account within the Boussinesq approximation
- More realistic boundary conditions are considered – horizontal layer

# Motivation

**Is it possible to extend these notions to Convective Hydromagnetic (CHM) systems?**

- The velocity field obeys the Navier-Stokes equation with Lorentz force
- Convection is taken into account within the Boussinesq approximation
- More realistic boundary conditions are considered – horizontal layer

**Is it possible to apply multiscale analysis to these systems?**

**Is it possible to find negative eddy diffusivity leading to the increase of magnetic field?**

# Equations describing a CHM System

- Navier-Stokes equation + Lorentz force + buoyancy force

$$\partial_t \mathbf{V} = \mathbf{V} \times (\partial \times \mathbf{V}) - \partial p + \nu \partial^2 \mathbf{V}$$
$$- \frac{1}{\rho_0 \mu_0} \mathbf{B} \times (\partial \times \mathbf{B}) - \alpha (T - T_0) \mathbf{G} + \mathbf{F}^{\mathbf{V}},$$

# Equations describing a CHM System

- Navier-Stokes equation + Lorentz force + buoyancy force

$$\partial_t \mathbf{V} = \mathbf{V} \times (\partial \times \mathbf{V}) - \partial p + \nu \partial^2 \mathbf{V} - \frac{1}{\rho_0 \mu_0} \mathbf{B} \times (\partial \times \mathbf{B}) - \alpha (T - T_0) \mathbf{G} + \mathbf{F}^{\mathbf{V}},$$

- Induction equation

$$\partial_t \mathbf{B} = \partial \times (\mathbf{V} \times \mathbf{B}) + \eta \partial^2 \mathbf{B} + \mathbf{F}^{\mathbf{B}}$$

# Equations describing a CHM System

- Navier-Stokes equation + Lorentz force + buoyancy force

$$\partial_t \mathbf{V} = \mathbf{V} \times (\partial \times \mathbf{V}) - \partial p + \nu \partial^2 \mathbf{V}$$
$$- \frac{1}{\rho_0 \mu_0} \mathbf{B} \times (\partial \times \mathbf{B}) - \alpha (T - T_0) \mathbf{G} + \mathbf{F}^{\mathbf{V}},$$

- Induction equation

$$\partial_t \mathbf{B} = \partial \times (\mathbf{V} \times \mathbf{B}) + \eta \partial^2 \mathbf{B} + \mathbf{F}^{\mathbf{B}}$$

- Heat transfer equation (with Joule term)

$$\partial_t T = -(\mathbf{V} \cdot \partial) T + k \partial^2 T + \frac{\eta}{c \rho_0 \mu_0} |\partial \times \mathbf{B}|^2 + F^T$$

# Equations describing a CHM System

- Navier-Stokes equation + Lorentz force + buoyancy force

$$\partial_t \mathbf{V} = \mathbf{V} \times (\partial \times \mathbf{V}) - \partial p + \nu \partial^2 \mathbf{V}$$
$$- \frac{1}{\rho_0 \mu_0} \mathbf{B} \times (\partial \times \mathbf{B}) - \alpha (T - T_0) \mathbf{G} + \mathbf{F}^{\mathbf{V}},$$

- Induction equation

$$\partial_t \mathbf{B} = \partial \times (\mathbf{V} \times \mathbf{B}) + \eta \partial^2 \mathbf{B} + \mathbf{F}^{\mathbf{B}}$$

- Heat transfer equation (with Joule term)

$$\partial_t T = -(\mathbf{V} \cdot \partial) T + k \partial^2 T + \frac{\eta}{c \rho_0 \mu_0} |\partial \times \mathbf{B}|^2 + F^T$$

- Solenoidality conditions

$$\partial \cdot \mathbf{V} = 0 \qquad \qquad \partial \cdot \mathbf{B} = 0$$

# Boundary conditions

- Horizontal layer $[0, L_1] \times [0, L_2] \times [0, L_3]$
- Periodicity in horizontal directions ($x_1$ and $x_2$)
- Boundary conditions along $x_3$:

$$
\begin{aligned}
\text{Velocity:} \quad & V_3\big|_{x_3=0,L_3} = 0 \\
& \partial_3 V_1\big|_{x_3=0,L_3} = \partial_3 V_2\big|_{x_3=0,L_3} = 0 \\
\\
\text{Magnetic Field:} \quad & B_3\big|_{x_3=0,L_3} = 0 \\
& \partial_3 B_1\big|_{x_3=0,L_3} = \partial_3 B_2\big|_{x_3=0,L_3} = 0 \\
\\
\text{Temperature:} \quad & T\big|_{x_3=0} = T_1 \\
& T\big|_{x_3=L_3} = T_2
\end{aligned}
$$

# Basic equations

## Change of variable:

$$T \rightsquigarrow \theta : \ \theta(x_3 = 0, \pi) = 0 \quad \Rightarrow \quad T(\theta) = \theta + T_1 + \delta T x_3; \quad \delta T = T_2 - T_1$$

$$\begin{cases} \partial_t \mathbf{V} - \mathbf{V} \times (\partial \times \mathbf{V}) = -\partial p + \nu \partial^2 \mathbf{V} - \mathbf{B} \times (\partial \times \mathbf{B}) - \alpha(\theta - \theta_0)\mathbf{G} + \mathbf{F} \\ \partial \cdot \mathbf{V} = 0 \\ \partial_t \mathbf{B} = \partial \times (\mathbf{V} \times \mathbf{B}) + \eta \partial^2 \mathbf{B} + \mathbf{R} \\ \partial \cdot \mathbf{B} = 0 \\ \partial_t T\theta + (\mathbf{V} \cdot \partial)\theta + \delta T V_3 = k \partial^2 \theta + \frac{\sigma}{2} \left| \partial \times \mathbf{B} \right|^2 + S \end{cases}$$

The velocity advection term has been replaced by it curl representation and the remaining gradient included in the pressure term (modified pressure)

# Symmetries

- **Parity-invariance**(vector or scalar field $\mathbf{f}$)**:**

| Parity-invariant | Parity-anti-invariant |
|---|---|
| $\mathbf{f}(-\mathbf{x}) = -\mathbf{f}(\mathbf{x})$ | $\mathbf{f}(-\mathbf{x}) = \mathbf{f}(\mathbf{x})$ |

# Symmetries

- **Parity-invariance**(vector or scalar field $\mathbf{f}$):

| Parity-invariant | Parity-anti-invariant |
|---|---|
| $\mathbf{f}(-\mathbf{x}) = -\mathbf{f}(\mathbf{x})$ | $\mathbf{f}(-\mathbf{x}) = \mathbf{f}(\mathbf{x})$ |

- **Symmetry about $x_3$ axis**(scalar field $s$):

| Symmetric | Anti-symmetric |
|---|---|
| $s(-x_1, -x_2, x_3) = s(x_1, x_2, x_3)$ | $s(-x_1, -x_2, x_3) = -s(x_1, x_2, x_3)$ |

- **Symmetry about $x_3$ axis**(vector field $\mathbf{Q}$):

| Symmetric | Anti-symmetric |
|---|---|
| $Q_1(-x_1, -x_2, x_3) = -Q_1(x_1, x_2, x_3)$<br>$Q_2(-x_1, -x_2, x_3) = -Q_2(x_1, x_2, x_3)$<br>$Q_3(-x_1, -x_2, x_3) = Q_3(x_1, x_2, x_3)$ | $Q_1(-x_1, -x_2, x_3) = Q_1(x_1, x_2, x_3)$<br>$Q_2(-x_1, -x_2, x_3) = Q_2(x_1, x_2, x_3)$<br>$Q_3(-x_1, -x_2, x_3) = -Q_3(x_1, x_2, x_3)$ |

# Symmetries consistent with the basic equations

- **V** and **B** parity-invariant
  - Navier-Stokes equation $\Rightarrow$ *p* parity-anti invariant and $\theta$ parity-invariant
  - Heat equation $\Rightarrow$ ohmic dissipation term must be neglected ($\sigma = 0$)

# Symmetries consistent with the basic equations

- **V** and **B** parity-invariant
  - Navier-Stokes equation $\Rightarrow$ $p$ parity-anti invariant and $\theta$ parity-invariant
  - Heat equation $\Rightarrow$ ohmic dissipation term must be neglected $(\sigma = 0)$
- **V** and **B** are symmetric about $x_3$ axis
  - Heat equation with ohmic dissipation $(\sigma \neq 0) \Rightarrow \theta$ must be a symmetric scalar field
  - Navier-Stokes equation $\Rightarrow$ $p$ must be a symmetric scalar field and $\mathbf{G} = (0, 0, -g)$

# Symmetries consistent with the basic equations

- **V** and **B** parity-invariant
  - Navier-Stokes equation $\Rightarrow$ $p$ parity-anti invariant and $\theta$ parity-invariant
  - Heat equation $\Rightarrow$ ohmic dissipation term must be neglected ($\sigma = 0$)

- **V** and **B** are symmetric about $x_3$ axis
  - Heat equation with ohmic dissipation ($\sigma \neq 0$) $\Rightarrow$ $\theta$ must be a symmetric scalar field
  - Navier-Stokes equation $\Rightarrow$ $p$ must be a symmetric scalar field and $\mathbf{G} = (0, 0, -g)$

- Anti-symmetric fields are inconsistent with the basic equations.
- The set of fields $p$, **V**, **B**, $\theta$ is called **symmetric** if it satisfies one of the symmetries defined above, and **anti-symmetric** if it satisfies the opposite symmetry.

# Multiscale Analysis

- Consider small perturbations to steady state solutions and obtain the linearised equations
- Consider new spatial ($\mathbf{X}$) and temporal ($t_L$) variables, describing large-scale dynamics

$$\mathbf{X} = \varepsilon \mathbf{x} \qquad\qquad t_L = \varepsilon_L t$$

- Assume that the fields depend on both fast ($\mathbf{x}$, $t$) and slow ($\mathbf{X}$, $t_L$) variables, as independent variables

# Multiscale Analysis

- Consider small perturbations to steady state solutions and obtain the linearised equations
- Consider new spatial ($\mathbf{X}$) and temporal ($t_L$) variables, describing large-scale dynamics

$$\mathbf{X} = \varepsilon \mathbf{x} \qquad\qquad t_L = \varepsilon_L t$$

- Assume that the fields depend on both fast ($\mathbf{x}$, $t$) and slow ($\mathbf{X}$, $t_L$) variables, as independent variables
- Expand the fields and derivatives in power series in the scaling parameters ($\varepsilon$, $\varepsilon_T$)
- Equate the terms in powers of $\varepsilon$ and $\varepsilon_T$ to obtain an hierarchy of equations

# Multiscale Analysis

- Consider small perturbations to steady state solutions and obtain the linearised equations
- Consider new spatial ($\mathbf{X}$) and temporal ($t_L$) variables, describing large-scale dynamics

$$\mathbf{X} = \varepsilon \mathbf{x} \qquad\qquad t_L = \varepsilon_L t$$

- Assume that the fields depend on both fast ($\mathbf{x}$, $t$) and slow ($\mathbf{X}$, $t_L$) variables, as independent variables
- Expand the fields and derivatives in power series in the scaling parameters ($\varepsilon$, $\varepsilon_T$)
- Equate the terms in powers of $\varepsilon$ and $\varepsilon_T$ to obtain an hierarchy of equations
- The equations in this hierarchy control the short-scale behaviour
- The solvability conditions control the large-scale behaviour

# Linearisation for a small perturbation

- Steady state solution of the CHM system: $\tilde{p}$, $\tilde{\mathbf{V}}$, $\tilde{\mathbf{B}}$, $\tilde{\theta}$
- Small perturbations: $pe^{\lambda t}$, $\mathbf{V}e^{\lambda t}$, $\mathbf{B}e^{\lambda t}$, $\theta e^{\lambda t}$
- $p \longrightarrow \tilde{p}+pe^{\lambda t}$, $\mathbf{V} \longrightarrow \tilde{\mathbf{V}}+\mathbf{V}e^{\lambda t}$, $\mathbf{B} \longrightarrow \tilde{\mathbf{B}}+\mathbf{B}e^{\lambda t}$, $\theta \longrightarrow \tilde{\theta}+\theta e^{\lambda t}$

$$\begin{cases} \nu\partial^2\mathbf{V} + \tilde{\mathbf{V}} \times (\partial \times \mathbf{V}) + \mathbf{V} \times (\partial \times \tilde{\mathbf{V}}) - \tilde{\mathbf{B}} \times (\partial \times \mathbf{B}) - \mathbf{B} \times (\partial \times \tilde{\mathbf{B}}) - \alpha\mathbf{G}\theta = \lambda\mathbf{V} + \partial p, \\ -\partial \times (\tilde{\mathbf{B}} \times \mathbf{V}) + \eta\partial\mathbf{B}^2 + \partial \times (\tilde{\mathbf{V}} \times \mathbf{B}) = \lambda\mathbf{B}, \\ -(\mathbf{V} \cdot \partial)\tilde{\theta} - \delta TV_3 + \sigma(\partial \times \tilde{\mathbf{B}}) \cdot (\partial \times \mathbf{B}) + k\partial^2\theta - (\tilde{\mathbf{V}} \cdot \partial)\theta = \lambda\theta, \\ \partial \cdot \mathbf{V} = 0, \\ \partial \cdot \mathbf{B} = 0. \end{cases}$$

- $\lambda$ is the growth rate of perturbations

# Block Notation (Dubrulle & Frisch)

$$\begin{cases} \mathbf{A}\mathbf{W} = \lambda \mathbf{W} + \begin{bmatrix} \partial p \\ \mathbf{0} \\ 0 \end{bmatrix} \\ \partial \cdot \mathbf{V} = 0 \\ \partial \cdot \mathbf{B} = 0 \end{cases} \qquad \mathbf{W} = \begin{bmatrix} \mathbf{V} \\ \mathbf{B} \\ \theta \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} \nu\partial^2 + \tilde{\mathbf{V}} \times (\partial \times \bullet) - (\partial \times \tilde{\mathbf{V}})\times & -\tilde{\mathbf{B}} \times (\partial \times \bullet) + (\partial \times \tilde{\mathbf{B}})\times & -\alpha \mathbf{G} \\ -\partial \times (\tilde{\mathbf{B}} \times \bullet) & \eta\partial^2 + \partial \times (\tilde{\mathbf{V}} \times \bullet) & 0 \\ -(\bullet \cdot \partial)\tilde{\theta} - \delta T \mathbf{e}_3 \cdot & \sigma(\partial \times \tilde{\mathbf{B}}) \cdot (\partial \times \bullet) & k\partial^2 - \tilde{\mathbf{V}} \cdot \partial \end{bmatrix}$$

- **A** preserves the symmetry of both symmetric and antisymmetric fields.

# The two-scales expansion

- Introduce large scale or slow variables ($\mathbf{X} = \varepsilon \mathbf{x}$) in the horizontal directions
- Assume $\mathbf{W} = \mathbf{W}(\mathbf{x}, \mathbf{X})$ and $p = p(\mathbf{x}, \mathbf{X})$
- Expand $\mathbf{W}$, $p$ and $\lambda$ in a power series of $\varepsilon$:

$$\mathbf{W} = \mathbf{W}^{(0)} + \varepsilon \mathbf{W}^{(1)} + \varepsilon^2 \mathbf{W}^{(2)} + \cdots + \varepsilon^n \mathbf{W}^{(n)} + O(\varepsilon^{n+1})$$
$$p = p^{(0)} + \varepsilon p^{(1)} + \varepsilon^2 p^{(2)} + \cdots + \varepsilon^n p^{(n)} + O(\varepsilon^{n+1})$$
$$\lambda = \lambda_0 + \varepsilon \lambda_1 + \varepsilon^2 \lambda_2 + \cdots + \varepsilon^n \lambda_n + O(\varepsilon^{n+1})$$

- Derivatives $\partial_i$ in $\mathbf{A}$ must be replaced by $\partial_i + \varepsilon \nabla_i$ ($\nabla_i = \frac{\partial}{\partial \mathbf{X}_i}$), hence $\mathbf{A} = \mathbf{A}^{(0)} + \varepsilon \mathbf{A}^{(1)} + \varepsilon^2 \mathbf{A}^{(2)}$

# The two-scales expansion

$$\mathbf{A}^{(0)} = \left[ \begin{array}{ccc} \nu\partial^2 + \tilde{\mathbf{V}} \times (\partial \times \bullet) - (\partial \times \tilde{\mathbf{V}})\times & -\tilde{\mathbf{B}} \times (\partial \times \bullet) + (\partial \times \tilde{\mathbf{B}})\times & -\alpha\mathbf{G} \\ -\partial \times (\tilde{\mathbf{B}} \times \bullet) & \eta\partial^2 + \partial \times (\tilde{\mathbf{V}} \times \bullet) & 0 \\ -(\bullet \cdot \partial)\tilde{\theta} - \delta T \mathbf{e}_3 \cdot & \sigma(\partial \times \tilde{\mathbf{B}}) \cdot (\partial \times \bullet) & k\partial^2 - \tilde{\mathbf{V}} \cdot \partial \end{array} \right] \equiv \mathbf{A}$$

$$\mathbf{A}^{(1)} = \left[ \begin{array}{ccc} 2\nu\partial \cdot \nabla + \tilde{\mathbf{V}} \times (\nabla \times \bullet) & -\tilde{\mathbf{B}} \times (\nabla \times \bullet) & 0 \\ -\nabla \times (\tilde{\mathbf{B}} \times \bullet) & 2\eta\partial \cdot \nabla + \nabla \times (\tilde{\mathbf{V}} \times \bullet) & 0 \\ 0 & \sigma(\partial \times \tilde{\mathbf{B}}) \cdot (\nabla \times \bullet) & 2k\partial \cdot \nabla - \tilde{\mathbf{V}} \cdot \nabla \end{array} \right]$$

$$\mathbf{A}^{(2)} = \left[ \begin{array}{ccc} \nu\nabla^2 & 0 & 0 \\ 0 & \eta\nabla^2 & 0 \\ 0 & 0 & k\nabla^2 \end{array} \right] = \mathbf{\Xi}\nabla^2, \quad \mathbf{\Xi} = \left[ \begin{array}{ccc} \nu & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & k \end{array} \right]$$

- $\mathbf{A}^{(0)}$ and $\mathbf{A}^{(2)}$ preserve the symmetries of both symmetric and anti-symmetric fields
- $\mathbf{A}^{(1)}$ exchanges the symmetry of both symmetric and anti-symmetric fields, since$\nabla_3 = 0$

# Hierarchy of Equations

$$
\begin{aligned}
\text{order } 0: \quad & \mathbf{A}^{(0)}\mathbf{W}^{(0)} = \lambda_0 \mathbf{W}^{(0)} + \begin{bmatrix} \partial p^{(0)} \\ 0 \\ 0 \end{bmatrix} \\
& \partial \cdot \mathbf{V}^{(0)} = 0 \\
& \partial \cdot \mathbf{B}^{(0)} = 0 \\
\text{order } 1: \quad & \mathbf{A}^{(0)}\mathbf{W}^{(1)} = -\mathbf{A}^{(1)}\mathbf{W}^{(0)} \\
& + \lambda_0 \mathbf{W}^{(1)} + \lambda_1 \mathbf{W}^{(0)} + \begin{bmatrix} \partial p^{(1)} + \nabla p^{(0)} \\ 0 \\ 0 \end{bmatrix} \\
& \partial \cdot \mathbf{V}^{(1)} + \nabla \cdot \mathbf{V}^{(0)} = 0 \\
& \partial \cdot \mathbf{B}^{(1)} + \nabla \cdot \mathbf{B}^{(0)} = 0
\end{aligned}
$$

# Hierarchy of Equations

$$
\begin{aligned}
\text{order 2:}\quad & \mathbf{A}^{(0)}\mathbf{W}^{(2)} = -\mathbf{A}^{(1)}\mathbf{W}^{(1)} - \mathbf{A}^{(2)}\mathbf{W}^{(0)} \\
& +\lambda_0\mathbf{W}^{(2)} + \lambda_1\mathbf{W}^{(1)} + \lambda_2\mathbf{W}^{(0)} + \begin{bmatrix} \partial p^{(2)} + \nabla p^{(1)} \\ 0 \\ 0 \end{bmatrix} \\
& \partial \cdot \mathbf{V}^{(2)} + \nabla \cdot \mathbf{V}^{(1)} = 0 \\
& \partial \cdot \mathbf{B}^{(2)} + \nabla \cdot \mathbf{B}^{(1)} = 0 \\
& \vdots \\
\text{order } n:\quad & \mathbf{A}^{(0)}\mathbf{W}^{(n)} = -\mathbf{A}^{(1)}\mathbf{W}^{(n-1)} - \mathbf{A}^{(2)}\mathbf{W}^{(n-2)} \\
& +\sum_{i=0}^{n} \lambda_i \mathbf{W}^{(n-i)} + \begin{bmatrix} \partial p^{(n)} + \nabla p^{(n-1)} \\ 0 \\ 0 \end{bmatrix} \\
& \partial \cdot \mathbf{V}^{(n)} + \nabla \cdot \mathbf{V}^{(n-1)} = 0 \\
& \partial \cdot \mathbf{B}^{(n)} + \nabla \cdot \mathbf{B}^{(n-1)} = 0
\end{aligned}
$$

# Large-scale fields

- Let $\mathbf{F} = \mathbf{F}(\mathbf{x}, \mathbf{X})$ and $\mathbf{G} = \mathbf{G}(\mathbf{x}, \mathbf{X})$ be scalar or vector fields:
  - The **average** over fast variables $(\mathbf{x})$, $\langle \mathbf{F} \rangle = \frac{1}{\mathcal{V}} \int_{\mathcal{V}} \mathbf{F} \, d\mathcal{V}$, depends only on slow variables $(\mathbf{X})$
  - $\langle \mathbf{F} \rangle$ is called the **large-scale component** of the field $\mathbf{F}$
  - The **fluctuation** of the field $\mathbf{F}$, $\{\mathbf{F}\} \overset{\text{def}}{=} \mathbf{F} - \langle \mathbf{F} \rangle$, depends both on fast and slow variables
  - The $\mathcal{L}^2$ **scalar product**, $\langle \mathbf{F}, \mathbf{G} \rangle = \int_{\mathcal{V}} \mathbf{F}^* \cdot \mathbf{G} \, d\mathcal{V}$, depends only on slow variables $(\mathbf{X})$

# Large-scale fields

- Let $\mathbf{F} = \mathbf{F}(\mathbf{x}, \mathbf{X})$ and $\mathbf{G} = \mathbf{G}(\mathbf{x}, \mathbf{X})$ be scalar or vector fields:
  - The **average** over fast variables ($\mathbf{x}$), $\langle \mathbf{F} \rangle = \frac{1}{\mathcal{V}} \int_{\mathcal{V}} \mathbf{F} \, d\mathcal{V}$, depends only on slow variables ($\mathbf{X}$)
  - $\langle \mathbf{F} \rangle$ is called the **large-scale component** of the field $\mathbf{F}$
  - The **fluctuation** of the field $\mathbf{F}$, $\{\mathbf{F}\} \overset{\text{def}}{=} \mathbf{F} - \langle \mathbf{F} \rangle$, depends both on fast and slow variables
  - The $\mathcal{L}^2$ **scalar product**, $\langle \mathbf{F}, \mathbf{G} \rangle = \int_{\mathcal{V}} \mathbf{F}^* \cdot \mathbf{G} \, d\mathcal{V}$, depends only on slow variables ($\mathbf{X}$)

- The large-scale components of $\mathbf{V}^{(n)}$ and $\mathbf{B}^{(n)}$ are solenoidal with respect to large scale variables:

$$\nabla \cdot \langle \mathbf{V}^{(n)} \rangle = 0 \qquad \nabla \cdot \langle \mathbf{B}^{(n)} \rangle = 0$$

# General remarks on solvability

- Problems to solve:

$$\mathbf{PA}^{(0)}\mathbf{PF} = \mathbf{PG}$$

- Projection operator:

$$\mathbf{P}\begin{bmatrix} \mathbf{Q^V} \\ \mathbf{Q^B} \\ Q^\theta \end{bmatrix} = \begin{bmatrix} \mathbf{Q^V} - \partial Q^{Vp} \\ \mathbf{Q^B} - \partial Q^{Hp} \\ Q^\theta \end{bmatrix}, \quad \partial^2 Q^{Vp} = \partial\cdot\left(\mathbf{A}^{(0)}\mathbf{Q}\right)^{\mathbf{V}}, \quad \partial^2 Q^{Hp} = \partial\cdot\left(\mathbf{A}^{(0)}\mathbf{Q}\right)^{H},$$

# General remarks on solvability

- Problems to solve:

$$\mathbf{PA}^{(0)}\mathbf{PF} = \mathbf{PG}$$

- Projection operator:

$$\mathbf{P}\begin{bmatrix} \mathbf{Q}^{\mathbf{V}} \\ \mathbf{Q}^{\mathbf{B}} \\ Q^{\theta} \end{bmatrix} = \begin{bmatrix} \mathbf{Q}^{\mathbf{V}} - \partial Q^{Vp} \\ \mathbf{Q}^{\mathbf{B}} - \partial Q^{Hp} \\ Q^{\theta} \end{bmatrix}, \quad \partial^2 Q^{Vp} = \partial \cdot \left(\mathbf{A}^{(0)}\mathbf{Q}\right)^{\mathbf{V}}, \quad \partial^2 Q^{Hp} = \partial \cdot \left(\mathbf{A}^{(0)}\mathbf{Q}\right)^{H},$$

- **Solvability:** a solution **F** exists if and only if **G** is orthogonal to $\ker\left(\mathbf{PA}^{(0)}\mathbf{P}\right)^*$

$$\left(\mathbf{PA}^{(0)}P\right)^* = \mathbf{PA}^{(0)^*}\mathbf{P}$$

$$\mathbf{A}^{(0)*} = \begin{bmatrix} \nu\partial^2 - \partial\times(\tilde{\mathbf{V}}\times\bullet) + (\partial\times\tilde{\mathbf{V}})\times & \tilde{\mathbf{B}}\times(\partial\times\bullet) & -\mathbf{e}_3\delta T - (\partial\tilde{\theta}) \\ \partial\times(\tilde{\mathbf{B}}\times\bullet) - (\partial\times\tilde{\mathbf{B}})\times & \eta\partial^2 - \tilde{\mathbf{V}}\times(\partial\times\bullet) & \begin{array}{c}\sigma\partial\times(\partial\times\tilde{\mathbf{B}}) \\ -\sigma(\partial\times\tilde{\mathbf{B}})\times(\partial\bullet)\end{array} \\ \alpha\mathbf{G}\cdot & 0 & k\partial^2 + \tilde{\mathbf{V}}\cdot\partial \end{bmatrix}$$

# Solution at order 0

- $\left\langle \mathbf{A}^{(0)}\mathbf{W}^{(0)} \right\rangle = 0$ (integration by parts and boundary conditions). Hence:
  - $\lambda_0 = 0$ (seeking solutions with $\langle \mathbf{V}^{(0)} \rangle$ and $\langle \mathbf{B}^{(0)} \rangle$ not vanishing simultaneously)
- Problem at order 0

$$\mathbf{A}^{(0)}\mathbf{W}^{(0)} = \begin{bmatrix} \partial p^{(0)} \\ 0 \\ 0 \end{bmatrix}$$

- By linearity:

$$\mathbf{W}^{(0)} = \sum_{i=1}^{m} a_i \mathbf{S}_i$$

$$p^{(0)} = \sum_{i=1}^{m} a_i S_i^p; \quad m = \dim(\ker \mathbf{A}^{(0)})$$

# Solution at order 0

$$\left\{\begin{array}{l} \mathbf{A}^{(0)}\mathbf{S}_i = \begin{bmatrix} \partial S_i^p \\ \mathbf{0} \\ 0 \end{bmatrix} \\ \partial \cdot \mathbf{S}_i^{\mathbf{V}} = 0 \\ \partial \cdot \mathbf{S}_i^{\mathbf{B}} = 0 \end{array}\right. \qquad ; \mathbf{S}_i = \begin{bmatrix} \mathbf{S}_i^{\mathbf{V}} \\ \mathbf{S}_i^{\mathbf{B}} \\ S_i^\theta \end{bmatrix}$$

$$\partial^2 S_i^p = \partial \cdot \left(\mathbf{A}^{(0)}\mathbf{S}_i\right)^{\mathbf{V}}$$

- Split $\mathbf{S}_i$ into average plus fluctuation
- $\mathbf{A}^{(0)}$ invertible in the subspace of mean-free fields $\Rightarrow$ the kernel must have a non-vanishing average part
- Boundary conditions $\Rightarrow$ only components $V_1,\ V_2,\ H_1,\ H_2$ can have non-vanishing average parts $\Rightarrow m = 4$

# Auxiliary problems at order 0

$$\langle \mathbf{S}_1 \rangle = \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{0} \\ 0 \end{bmatrix}, \quad \langle \mathbf{S}_2 \rangle = \begin{bmatrix} \mathbf{e}_2 \\ \mathbf{0} \\ 0 \end{bmatrix}, \quad \langle \mathbf{S}_3 \rangle = \begin{bmatrix} \mathbf{0} \\ \mathbf{e}_1 \\ 0 \end{bmatrix}, \quad \langle \mathbf{S}_4 \rangle = \begin{bmatrix} \mathbf{0} \\ \mathbf{e}_2 \\ 0 \end{bmatrix}$$

$$\mathbf{P}\left(\mathbf{A}^{(0)}\{\mathbf{S}_1\} + \begin{bmatrix} \begin{bmatrix} 0 \\ \partial_2 \tilde{V}_1 - \partial_1 \tilde{V}_2 \\ \partial_3 \tilde{V}_1 - \partial_1 \tilde{V}_3 \end{bmatrix} \\ -\partial_1 \tilde{\mathbf{B}} \\ -\partial_1 \tilde{\theta} \end{bmatrix}\right) = 0 \qquad \mathbf{P}\left(\mathbf{A}^{(0)}\{\mathbf{S}_2\} + \begin{bmatrix} \begin{bmatrix} \partial_1 \tilde{V}_2 - \partial_2 \tilde{V}_1 \\ 0 \\ \partial_3 \tilde{V}_2 - \partial_2 \tilde{V}_3 \end{bmatrix} \\ -\partial_2 \tilde{\mathbf{B}} \\ -\partial_2 \tilde{\theta} \end{bmatrix}\right) = 0$$

$$\mathbf{P}\left(\mathbf{A}^{(0)}\{\mathbf{S}_3\} + \begin{bmatrix} \begin{bmatrix} 0 \\ \partial_1 \tilde{H}_2 - \partial_2 \tilde{H}_1 \\ \partial_1 \tilde{H}_3 - \partial_3 \tilde{V}_1 \end{bmatrix} \\ -\partial_1 \tilde{\mathbf{V}} \\ 0 \end{bmatrix}\right) = 0 \qquad \mathbf{P}\left(\mathbf{A}^{(0)}\{\mathbf{S}_4\} + \begin{bmatrix} \begin{bmatrix} \partial_2 \tilde{H}_1 - \partial_1 \tilde{H}_2 \\ 0 \\ \partial_2 \tilde{H}_3 - \partial_3 \tilde{H}_2 \end{bmatrix} \\ -\partial_2 \tilde{\mathbf{V}} \\ 0 \end{bmatrix}\right) = 0$$

# Solution at order 1

- $\left\langle \mathbf{A}^{(0)}\mathbf{W}^{(1)} \right\rangle = \left\langle \mathbf{A}^{(1)}\mathbf{W}^{(0)} \right\rangle = 0$ (integration by parts and boundary conditions). Hence:
  - $\lambda_1 = 0$ (seeking solutions with $\langle \mathbf{V}^{(0)} \rangle$ and $\langle \mathbf{B}^{(0)} \rangle$ not vanishing simultaneously)
  - $\langle p^{(0)} \rangle = 0$
- Problem at order 1

$$\begin{aligned}
\mathbf{A}^{(0)}\mathbf{W}^{(1)} &= -\mathbf{A}^{(1)}\mathbf{W}^{(0)} + \begin{bmatrix} \nabla p^{(0)} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \partial p^{(1)} \\ 0 \\ 0 \end{bmatrix} \\
&= \sum_{i=1}^{m}\sum_{j=1}^{2} \mathbf{M}_{ij}\nabla_j \alpha_i + \begin{bmatrix} \partial p^{(1)} \\ 0 \\ 0 \end{bmatrix} \quad ; \quad \mathbf{M}_{ij} = -\mathbf{B}_j \mathbf{S}_i + \begin{bmatrix} \mathbf{e}_j S_i^p \\ \mathbf{0} \\ 0 \end{bmatrix}
\end{aligned}$$

$$\mathbf{B}_j = \begin{bmatrix} 2\nu\partial_j + \mathbf{e}_j \tilde{\mathbf{V}} \cdot \bullet - \tilde{V}_j & -\mathbf{e}_j \tilde{\mathbf{B}} \cdot \bullet + \tilde{H}_j & 0 \\ -\tilde{\mathbf{B}}\mathbf{e}_j \cdot \bullet + \tilde{H}_j & 2\eta\partial_j + \tilde{\mathbf{V}}\mathbf{e}_j \cdot \bullet - \tilde{V}_j & 0 \\ 0 & \sigma \sum_{k=1}^{3} \left( \partial_j \tilde{H}_k - \partial_k \tilde{H}_j \right) \mathbf{e}_k \cdot & 2k\partial_j - \tilde{V}_j \end{bmatrix}.$$

# Solution at order 1

- By linearity:

$$\mathbf{W}^{(1)} = \sum_{i=1}^{m} \sum_{j=1}^{2} \nabla_j a_i \mathbf{\Gamma}_{ij} + \sum_{i=1}^{m} b_i \mathbf{S}_i$$

$$p^{(1)} = \sum_{i=1}^{m} \sum_{j=1}^{2} \nabla_j a_i \Gamma_{ij}^{p} + \sum_{i=1}^{m} b_i S_i^{p}$$

$$\left\{ \begin{array}{l} \mathbf{A}^{(0)} \mathbf{\Gamma}_{ij} = \mathbf{M}_{ij} + \begin{bmatrix} \partial \Gamma_{ij}^{p} \\ 0 \\ 0 \end{bmatrix} \\ \partial \cdot \mathbf{\Gamma}_{ij}^{\mathbf{V}} = -\left(S_i^{\mathbf{V}}\right)_j \\ \partial \cdot \mathbf{\Gamma}_{ij}^{\mathbf{B}} = -\left(S_i^{\mathbf{B}}\right)_j \end{array} \right. \quad ; \mathbf{\Gamma}_{ij} = \begin{bmatrix} \mathbf{\Gamma}_{ij}^{\mathbf{V}} \\ \mathbf{\Gamma}_{ij}^{\mathbf{B}} \\ \Gamma_{ij}^{\theta} \end{bmatrix}$$

$$\partial^2 \Gamma_{ij}^{p} = \partial \cdot \left( \mathbf{M}_{ij} + \mathbf{A}^{(0)} \mathbf{\Gamma}_{ij} \right)^{\mathbf{V}}$$

# Auxiliary problems at order 1

- 8 auxiliary problems at order 1:

$$\mathbf{P}\mathbf{\Gamma}'_{ij} = \mathbf{P}\left(\mathbf{M}_{ij} - \mathbf{A}^{(0)}\begin{bmatrix}\partial\Pi^{\mathbf{V}}_{ij}\\ \partial\Pi^{\mathbf{B}}_{ij}\\ 0\end{bmatrix}\right), \quad i = 1,\dots,4,\ j = 1,2$$

$$\mathbf{\Gamma}_{ij} = \mathbf{\Gamma}'_{ij} + \begin{bmatrix}\partial\Pi^{\mathbf{V}}_{ij}\\ \partial\Pi^{\mathbf{B}}_{ij}\\ 0\end{bmatrix},$$

$$\partial^2\Pi^{\mathbf{V}}_{ij} = -\left(S^{\mathbf{V}}_i\right)_j, \qquad \partial^2\Pi^{\mathbf{B}}_{ij} = -\left(S^{\mathbf{B}}_i\right)_j.$$

# Solvability at order 2 - closed equations for $\mathbf{W}^{(0)}$

- Orthogonality to $\ker \mathbf{PA}^{(0)*}\mathbf{P}$: $\;$ for any $\mathbf{C}_l \in \ker \mathbf{PA}^{(0)*}\mathbf{P}$,

$$\lambda_2 \sum_{i=1}^{4} \delta_{li} a_i + \sum_{i=1}^{4} \sum_{j,k=1}^{2} \langle \mathbf{C}_l, -\mathbf{\Xi S}_i \delta_{jk} - \mathbf{B}_k \mathbf{\Gamma}_{ij} \rangle \nabla_k \nabla_j a_i + \langle \mathbf{C}_l^{\mathbf{V}}, \nabla \langle p^{(1)} \rangle \rangle = 0$$

- Second order equation with constant coefficients $\longrightarrow$ admits Fourier harmonics as solutions

$$a_i(\mathbf{X}) = \hat{a}_i(\mathbf{q}) e^{i\mathbf{q}\cdot\mathbf{X}}; \quad \mathbf{q} \text{ arbitrary wave vector}$$

- Eigenvalue equation for Fourier modes (4x4 eigenvalue problem):

$$\lambda_2 \begin{bmatrix} \hat{a}_1 + \nu \\ \hat{a}_2 + \nu \\ \hat{a}_3 + \eta \\ \hat{a}_4 + \eta \end{bmatrix} + \mathbf{E} \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \\ \hat{a}_4 \end{bmatrix} = -i\hat{p}(\mathbf{q}) \begin{bmatrix} q_1 \\ q_2 \\ 0 \\ 0 \end{bmatrix}$$

$$\hat{a}_1 q_1 + \hat{a}_2 q_2 = 0$$

$$\hat{a}_3 q_1 + \hat{a}_4 q_2 = 0$$

$$\mathbf{E}_{li} = \sum_{j,k=1}^{2} q_k q_j \langle \mathbf{C}_l, -\mathbf{B}_k \mathbf{\Gamma}_{ij} \rangle$$

# The large-scale eigenvalue equation

$$\begin{bmatrix} (\lambda_2+\nu)\hat{a}'_1 \\ (\lambda_2+\eta)\hat{a}'_2 \end{bmatrix} + \mathbf{E}' \begin{bmatrix} \hat{a}'_1 \\ \hat{a}'_2 \end{bmatrix} = 0$$

with

$$\begin{aligned}
\mathbf{E}'_{11} &= \mathbf{E}_{11} q_2^2 - (\mathbf{E}_{12} + \mathbf{E}_{21}) q_1 q_2 + \mathbf{E}_{22} q_1^2, \\
\mathbf{E}'_{12} &= \mathbf{E}_{13} q_2^2 - (\mathbf{E}_{14} + \mathbf{E}_{23}) q_1 q_2 + \mathbf{E}_{24} q_1^2, \\
\mathbf{E}'_{21} &= \mathbf{E}_{31} q_2^2 - (\mathbf{E}_{32} + \mathbf{E}_{41}) q_1 q_2 + \mathbf{E}_{42} q_1^2, \\
\mathbf{E}'_{22} &= \mathbf{E}_{33} q_2^2 - (\mathbf{E}_{34} + \mathbf{E}_{43}) q_1 q_2 + \mathbf{E}_{44} q_1^2.
\end{aligned}$$

$$\mathbf{q} = (\cos\vartheta, \sin\vartheta),\ \vartheta \in [0, 2\pi] \Rightarrow \lambda_2^{\pm}(\vartheta) = -\frac{b}{2}\left(1 \mp \sqrt{1 - \frac{4c}{b^2}}\right),$$

$$b = \nu + \eta + E'_{11} + E'_{22},\ \ c = \nu\eta + \nu E'_{22} + \eta E'_{11} + E'_{11}E'_{22} - E'_{12}E'_{21}$$

$$\lambda_2^{max} = \max_{\vartheta\in[0,2\pi]} \max\left\{\lambda_2^+(\vartheta), \lambda_2^-(\vartheta)\right\} \tag{1}$$

$$\lambda_2^{min} = \min_{\vartheta\in[0,2\pi]} \min\left\{\lambda_2^+(\vartheta), \lambda_2^-(\vartheta)\right\} \tag{2}$$

# Numerical Procedure

- Evaluate **AW** via Spectral methods:

$$f(x,y,z) = \sum_{n_{k_x} n_{k_y} n_{k_z}} \hat{f}(k_x,k_y,k_z)\mathbf{e}^{i(k_x x + k_y y)} \left| \begin{array}{c} \sin(k_z z) \\ \cos(k_z z) \end{array} \right|,$$

  with $k_x = \frac{2\pi}{L_1} n_{k_x}$, $k_y = \frac{2\pi}{L_2} n_{k_y}$, $k_z = \frac{\pi}{L_3} n_{k_z}$.
- In the Fourier space derivatives are replaced by multiplication with the wave vectors
- Products are evaluated in the real space (Pseudo-Spectral methods)

# Numerical Procedure

- Evaluate **AW** via Spectral methods:

$$f(x,y,z) = \sum_{n_{k_x} n_{k_y} n_{k_z}} \hat{f}(k_x,k_y,k_z)\mathbf{e}^{i(k_x x + k_y y)} \left| \begin{array}{c} \sin(k_z z) \\ \cos(k_z z) \end{array} \right|,$$

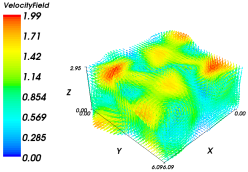
  with $k_x = \frac{2\pi}{L_1} n_{k_x},\ k_y = \frac{2\pi}{L_2} n_{k_y},\ k_z = \frac{\pi}{L_3} n_{k_z}.$
- In the Fourier space derivatives are replaced by multiplication with the wave vectors
- Products are evaluated in the real space (Pseudo-Spectral methods)

- Short-scale Stability
  - Find the dominant eigenvalue of **AW** using an algorithm based on Arnoldi's method.
- Large-scale Stability
  - Solve $\mathbf{AW} = \mathbf{g}$ (for each auxiliary problem), using, for instance the conjugate gradients method.
  - Evaluate $\mathbf{E}$, $\mathbf{E}'$ and maximise $\lambda_2$ numerically as a

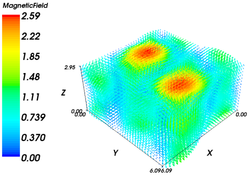
# Generating basic steady state fields

- Fourier coefficients are randomly generated (boundary conditions are enforced by the choice of basis).
- Symmetry conditions are applied and the gradient of vector fields removed.
- The fields are normalised to have decaying spectra:
  - Algebraic: $E(k) \sim k^{-\beta}$
  - Exponential: $E(k) \sim \exp(-\beta k)$
- The fields are normalised to have a root mean square average of 1.

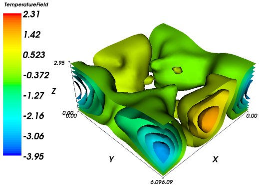
# Example of generated steady state fields: algebraic spectra



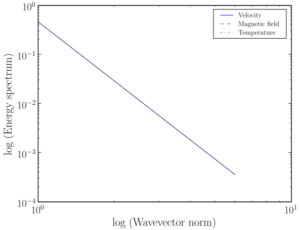

Steady state velocity




Steady state magnetic field

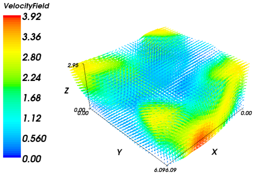



Steady state temperature

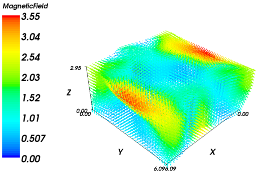



Steady state energy spectra

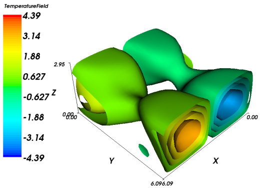
# $\mathbf{S}_1$: algebraic spectra



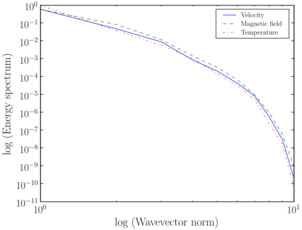

Steady state velocity




Steady state magnetic field

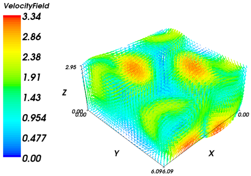



Steady state temperature

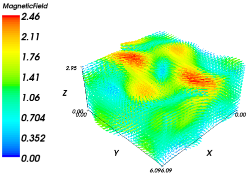



Steady state energy spectra

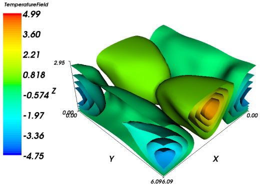
# $\Gamma_{11}$: algebraic spectra



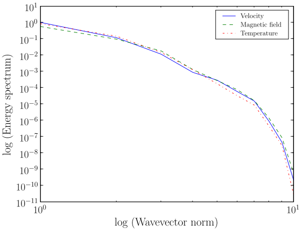

Steady state velocity




Steady state magnetic field

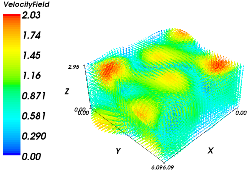



Steady state temperature

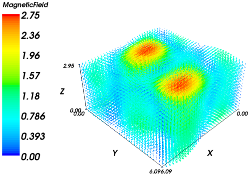



Steady state energy spectra

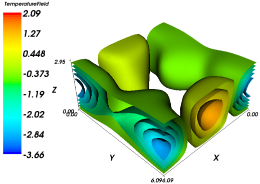
# Example of generated steady state fields: exponential spectra



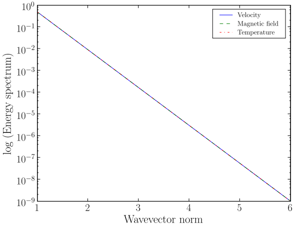

Steady state velocity




Steady state magnetic field

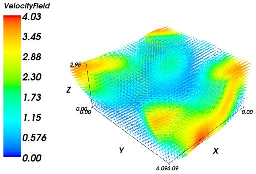



Steady state temperature

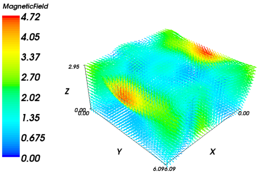



Steady state energy spectra

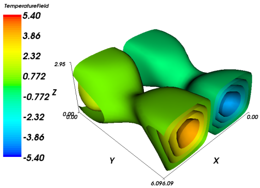
# $\mathbf{S}_1$: exponential spectra



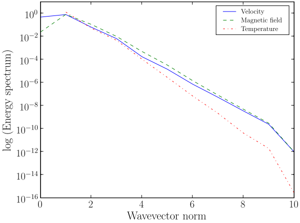

Steady state velocity




Steady state magnetic field

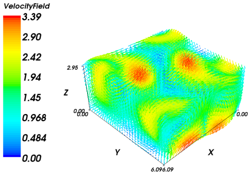



Steady state temperature

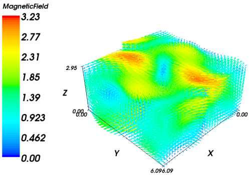



Steady state energy spectra

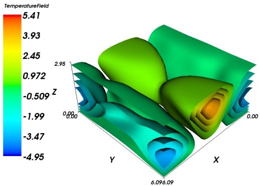
# $\Gamma_{11}$: exponential spectra



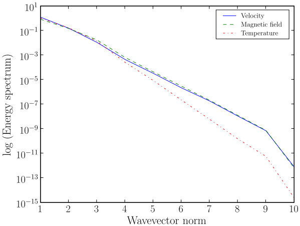

Steady state velocity




Steady state magnetic field




Steady state temperature




Steady state energy spectra

# Results

- Physical dimensions: $L_1 = 2\pi$, $L_2 = 2\pi$ and $L_3 = \pi$.
- Numerical grid: $32 \times 32 \times 16$.
- Basic fields: randomly generated with decaying algebraic spectra ($\beta = 4$, $0 \leq k < 5$ ) and symmetry about the $z-$ axis.
- Physical parameters: $\nu = 0.5$, $\eta = 0.3$ and $k = 0.5$,$\alpha = 1$, $g = -1$, $\delta T = -1$ and $\sigma = 0$.

| | Algebraic Spectra | Exponential Spectra |
|---|---|---|
| $\lambda_2^{min}$ | $-1.165$ | $-1.165$ |
| $\theta^{min}$ | 3.593 | 0.4511 |
| $\lambda_2^{max}$ | 1.426 | 1.426 |
| $\theta^{max}$ | 3.392 | 3.392 |
| $\lambda_{short}(S)$ | $-0.5662$ | $-0.56623$ |
| $\lambda_{short}(A)$ | $-0.05175$ | $-0.05638 + 0.04912\, i$ |

# Abundance of Negative Eddy Diffusivities

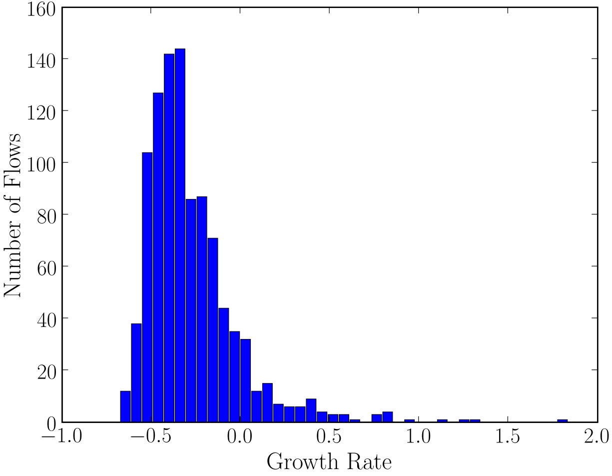



Figure: Statistics of growth rates for algebraic spectra: 11% positive values.

# Abundance of Negative Eddy Diffusivities

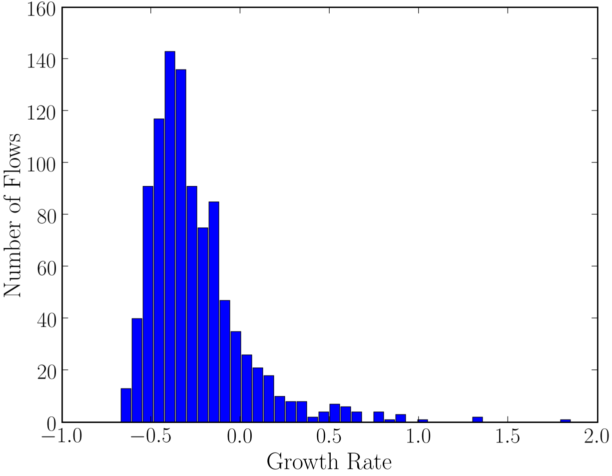



Figure: Statistics of growth rates (opposite of eddy diffusivity) for exponential spectra: 13% positive values.

# Growth Rates as a function of Molecular Diffusivities

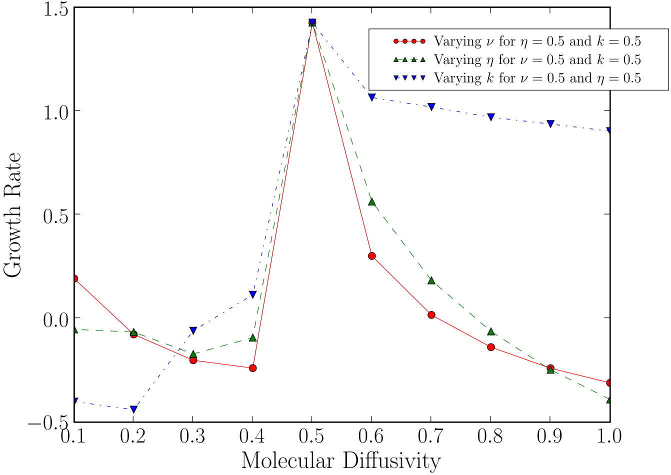



Figure: Algebraic spectra.

# Growth Rates as a function of Molecular Diffusivities

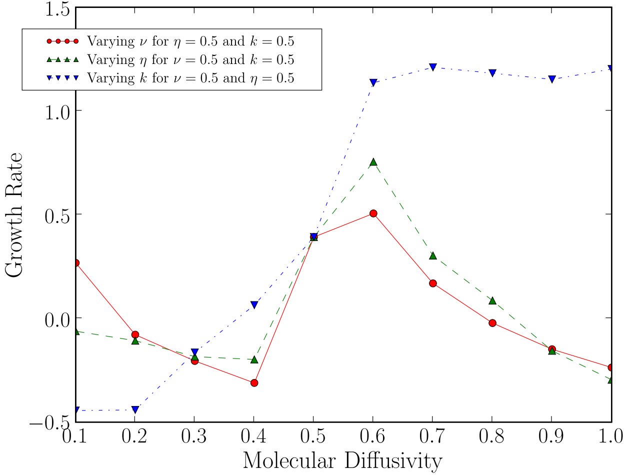



Figure: Exponential spectra.

# Summary

- It was possible to derive an equation for large-scale dominant modes of a CHM system, decoupling short and large-scale behaviour.
  - Symmetries are important to eliminate first order effects
  - Multiscale analysis of rotation is not straightforward

# Summary

- It was possible to derive an equation for large-scale dominant modes of a CHM system, decoupling short and large-scale behaviour.
  - Symmetries are important to eliminate first order effects
  - Multiscale analysis of rotation is not straightforward
- A code has been developed in C++ to solve the auxiliary problems and maximise $\lambda_2$.
  - Code tested against the results of asymptotic expansions for large molecular diffusivities.
  - Basic states: randomly generated fields with decaying energy spectrum, satisfying the required symmetries: only a statistical study is possible.

# Summary

- It was possible to derive an equation for large-scale dominant modes of a CHM system, decoupling short and large-scale behaviour.
  - Symmetries are important to eliminate first order effects
  - Multiscale analysis of rotation is not straightforward
- A code has been developed in C++ to solve the auxiliary problems and maximise $\lambda_2$.
  - Code tested against the results of asymptotic expansions for large molecular diffusivities.
  - Basic states: randomly generated fields with decaying energy spectrum, satisfying the required symmetries: only a statistical study is possible.
- There are short-scale stable steady states which the large-scale growth rate is positive, i.e. steady states that exhibit instability to large-scale perturbation: 11% for algebraic spectra and 13% for exponential spectra.

# Further Work

- Perform more runs
  - Further exploration of physical parameters
  - Explore the role of convection

# Further Work

- Perform more runs
  - Further exploration of physical parameters
  - Explore the role of convection

- Analytics
  - Perform expansions for different boundary conditions
  - Generalisation of expansion to higher orders
  - Include rotation

# Further Work

- Perform more runs
  - Further exploration of physical parameters
  - Explore the role of convection
- Analytics
  - Perform expansions for different boundary conditions
  - Generalisation of expansion to higher orders
  - Include rotation
- Codes
  - Generalise the spectral package GOOPS:
    - consider different basis of spectral function to support different geometries and boundary conditions
    - provide a more STL-like interface
  - Generalise the MHDC3DL code to solve the full non-linear problem